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MONTHLY NOTICES
OF THE
ROYAL ASTRONOMICAL SOCIETY.

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ROYAL ASTRONOMICAL SOCIETY.

VOL. LIX.

MAY 12, 1899.

No. 8.

Professor G. H. DARWIN, M.A., LL.D., President, in the Chair.

Rev. Edward Lyon Berthon, M.A., St. Margaret's, Cupernham, Romsey, Hants; and

Rev. Theodore Evelyn Reece Philips, M.A., Handford Vicarage, Yeovil, Somerset,

were balloted for and duly elected Fellows of the Society.

The following candidates were proposed for election as Fellows of the Society, the names of the proposers from personal knowledge being appended :—

Rev. Thomas Gerrard Barber, B.A., 10 Highfield Road, Doncaster (proposed by J. W. L. Glaisher); and
Sydney Samuel Hough, B.A., Chief Assistant, Royal Observatory, Cape of Good Hope (proposed by David Gill).

The following were proposed by the Council as Associates of the Society :—

G. E. Hale, D.Sc., F.R.A.S., Director of the Yerkes Observatory, Williams Bay, Wisconsin, U.S.A.;
F. R. Helmert, Director of the Geodetic Institute, Potsdam, Germany;

F. Küstner, Director of the Observatory, Bonn, Germany ;
and
Juan M. Thome, Director of the Argentine National Observatory, Cordoba, Argentine Republic.

Eighty-two presents were announced as having been received since the last meeting, including, amongst others :—

G. H. Darwin, *The Tides*, presented by the author ; Bordeaux Observatory, *Annales*, tome 8 ; and Munich Observatory, *Neue Annalen*, Band 3, presented by the Observatories.

On the Errors of Star Photographs due to Optical Distortion of the Object-glass with which the Photograph is taken. By
H. H. Turner, M.A., F.R.S., Savilian Professor.

1. When the International Conference assembled in 1887 to consider the construction of a chart of the heavens by photography, the most important question which met them at the outset was, With what kind of instrument should the work be done ? Possible instruments were the reflector, the refractor with achromatic object-glass, and the photographic doublet. The achromatism of the first-named, a great advantage in photographic work, was overbalanced by its strictly limited field, and certain inconveniences of working ; the large field of the photographic doublet was regarded with suspicion, suggesting possibilities of complex optical distortion ; and the simple photographic refractor was chosen. We have learnt a good deal more about stellar photography since 1887 ; and I would now venture the opinion that, though the choice of the simple refractor has been attended with many advantages, especially in the taking, measurement, and reduction of the "Catalogue" plates, the photographic doublet is the proper instrument for *charting* purposes. The fears about optical distortion are to a large extent groundless, as the present paper will show ; and the advantages of getting a large field are too obvious to need explanation. In this opinion I am, of course, only following at a respectful distance Professor E. C. Pickering, who urged the claims of the doublet on the Conference in 1887, and who has continued to work with various forms of the doublet ever since. But I have not seen published any attempt to show that the optical distortion of the doublet is small, and though the present investigation leaves much to be desired, it will to some extent fill this gap.

2. For examination of the optical distortion of a doublet Professor E. C. Pickering kindly sent to Oxford some positive copies of large plates (16 inches by 13 inches) taken at Arequipa with the Bruce doublet of the same focal length as the instruments used for the Astrographic Chart, viz., $11\frac{1}{2}$ feet, so that 1^{mm} on the plate corresponds to $1'$. A simple and direct way to examine the distortion would be to put a *réseau* on this plate (or on a copy of it) and measure the positions of known stars in different parts of the plate; but our measuring instruments, *réseaux*, &c., at Oxford are only adapted to plates of $2^\circ \times 2^\circ$, and this direct process is unfortunately not feasible. As an alternative, contact-negatives of different portions of the large positive were made on plates of the $2^\circ \times 2^\circ$ size, and compared with plates of exactly the same regions taken with the Oxford astrographic telescope. Thus we must first show that these latter plates are free from distortion.

The Oxford plates for the Astrographic Catalogue are free from optical distortion. §§ 3-II.

3. The process of measuring and reducing a catalogue plate has been several times described. Rectangular coordinates (x, y) of the stars on the plate are obtained by referring them to the *réseau*. If

(a) The focal length of the telescope were such that $1^{\text{mm}} = 1'$ exactly; and the telescope in exact adjustment;

(b) The plate were correctly oriented in the telescope;

(c) There were no refraction and aberration;

(d) The plate were flat, and normal to the line of collimation;

(e) The *réseau* were perfect;

(f) The lens had no optical distortion;

then these (x, y) coordinates would be the same as what have been called "standard coordinates" (ξ, η) connected with R.A. and N.P.D. by the formulæ

$$\mu\xi = \tan(\alpha - A) \sin q \sec(P - q), \quad \mu\eta = \tan(P - q). \quad (1)$$

where (α, p) are the R.A. and N.P.D. of the star,

(A, P) are the R.A. and N.P.D. of the plate-centre,

$$\tan q = \tan p \cos(\alpha - A),$$

and μ is the circular measure of $5'$.

4. Further, the sources of error indicated in (a), (b), (c) may be eliminated by assuming linear relations of the form

$$\xi = x + ax + by + c \quad \eta = y + dx + ey + f \quad (2)$$

For certain stars on the plate the R.A. and N.P.D., and hence ξ and η , are known; and when these have been measured so as to determine the x and y for them, from a set of equations such as (2), we can determine a, b, c, d, e, f . To determine these six constants, three known stars (giving three pairs of equations) are

theoretically enough ; but, owing to the existence of accidental errors, it is advisable to use as many stars as possible of which the R.A. and N.P.D. are known.

5. For the main part of the work assigned by the International Committee to the Oxford University Observatory, viz., zones $+25^\circ$ to $+31^\circ$, the "known" stars are taken from the meridian observations made at Cambridge in the years 1872 to 1896, in revision of Argelander's zones for the *Astronomische Gesellschaft* Catalogue. These observations give us an average of about 30 "known" stars on a plate, from which the constants a, b, c, d, e, f are determined, the actual process adopted being to group all the stars in the

N. half, S. half, E. half, W. half

of the plate, and take the mean in each case. Were the stars symmetrically distributed, this would give us eight equations of the form

$$\left. \begin{array}{ll} 0\cdot a + 19\cdot 5b + c = r_1 & 0\cdot d + 19\cdot 5e + f = R_1 \\ 0\cdot a + 6\cdot 5b + c = r_2 & 0\cdot d + 6\cdot 5e + f = R_2 \\ 19\cdot 5a + 0\cdot b + c = r_3 & 19\cdot 5d + 0\cdot e + f = R_3 \\ 6\cdot 5a + 0\cdot b + c = r_4 & 6\cdot 5d + 0\cdot e + f = R_4 \end{array} \right\} \dots \dots (3)$$

So that

$$\left. \begin{array}{ll} a = (r_3 - r_4)/13, & b = (r_1 - r_2)/13 \\ d = (R_3 - R_4)/13, & e = (R_1 - R_2)/13 \end{array} \right\} \dots \dots (4)$$

Substituting the values of a and b in each of the first set we get four values for c , which ought to agree if the arithmetic is correct (though the agreement is no check on the accuracy of the measures, for the four equations are not independent, and are really equivalent to three only).

In practice the stars are, of course, not symmetrically distributed, and the equations only approximate to this form ; but the method of solution is the same.

6. When a, b, c, d, e, f have been obtained, the values of $x + ax + by + c - \xi$ and $y + dx + ey + f - \eta$ are formed for the individual stars and called residuals. Now these residuals are affected with the errors indicated in $(d), (e), (f)$ above, viz. :

- (d) Curvature of the plate and tilt.
- (e) Errors of the *réseau*.
- (f) Optical distortion of the lens

as well as by

- (g) Errors of the meridian observations, and proper motions of the stars.

If we collect the residuals for a number of plates, and group them according to position of the star with reference to the centre of the plate, then errors (g) may be regarded as purely

accidental, and the same in all portions of the plate. It is probable also that the curvature of the plates may be treated as accidental in its effects, though of course it is not impossible that there may be a systematic curvature in all photographic plates leading to small systematic errors in the observations; but these would be very small indeed—the errors of errors—for in making the measures the curvature of the plate is corrected for in the correction for “runs”; and we can sensibly neglect errors from this cause.

7. Thus in the mean of a number of plates we should have the residuals affected systematically by

- (A) Optical distortion and tilt of the plate.
- (B) Errors of the *réseau*.

The latter might be separately determined, but this work has not yet been undertaken, as it is known that the *réseau* errors are very small. Moreover, in the measures collected below several different *réseaux* were used, and the errors would tend to compensate. The means given below may thus be regarded as showing optical distortion of the lens and tilt of the plate, if any.

8. The 40 plates examined were in $+26^{\circ}$, 18^h-24^h , and thus had a good many stars on them. They were grouped in hours of R.A., viz.: 8 plates in 18^h , 5 plates in 19^h , 7 in 20^h , 5 in 21^h , 8 in 22^h , 7 in 23^h .

Each plate was divided into 16 portions, exclusive of narrow strips enclosing the coordinate axes omitted for convenience; viz., each coordinate running from 0 to 26, the 16 portions were obtained by grouping together portions

$$0\ 0-6\ 0, 6\ 0-12\ 0, 14\ 0-20\ 0, 20\ 0-26\ 0$$

in each coordinate, so that the mean coordinates of the groups referred to the centre of the plate were approximately

$$\begin{array}{cccc} x = -10\ 0 & -4\ 0 & +4\ 0 & +10\ 0 \\ y = +10\ 0 & +10\ 0 & +10\ 0 & +10\ 0 \\ \\ x = -10\ 0 & -4\ 0 & +4\ 0 & +10\ 0 \\ y = +4\ 0 & +4\ 0 & +4\ 0 & +4\ 0 \\ \\ x = -10\ 0 & -4\ 0 & +4\ 0 & +10\ 0 \\ y = -4\ 0 & -4\ 0 & -4\ 0 & -4\ 0 \\ \\ x = -10\ 0 & -4\ 0 & +4\ 0 & +10\ 0 \\ y = -10\ 0 & -10\ 0 & -10\ 0 & -10\ 0 \end{array}$$

the strip $12\ 0$ to $14\ 0$ in each coordinate being omitted. The average number of stars in each group was 20. The groups are given separately as an indication of the probable error of a

determination. It will be noticed that the x residuals are larger than the y residuals, indicating that the Cambridge observations of N.P.D. are more accurate than those of R.A., for of course there is no difference in the photographic measures.

9. The unit adopted in the following table is $\cdot 0001$ of a *résseau* interval or $0''\cdot 03$. The largest mean residual, 24, thus represents $0''\cdot 72$.

Mean Residuals of Groups.

	In x .				In y .			
18 ^h	-13	+ 7	+13	- 2	- 1	0	- 1	5
19 ^h	+ 4	- 9	+ 8	-19	+ 1	+ 1	- 8	- 8
20 ^h	- 4	+12	- 8	- 9	- 6	- 2	- 3	+ 2
21 ^h	- 9	-14	-15	- 2	+ 3	+ 4	- 2	+ 6
22 ^h	-15	+ 1	- 4	- 9	- 6	- 2	- 8	+12
23 ^h	+ 8	+24	+ 1	+11	+15	-10	+26	+ 2
Mean	- 5	+ 4	- 1	- 5	+ 1	- 2	+ 1	+ 2
18 ^h	+ 8	-13	- 6	+11	- 5	- 7	- 4	- 2
19 ^h	+11	+16	+18	+ 5	+ 1	+14	+12	- 1
20 ^h	+ 2	+11	- 1	-24	- 1	- 3	- 3	+ 2
21 ^h	+15	+ 7	- 2	0	+ 2	- 8	- 4	- 2
22 ^h	+ 3	- 2	+10	- 2	- 1	- 8	-12	+ 7
23 ^h	- 3	-21	+15	- 4	-13	- 1	+ 3	-12
Mean	+ 6	0	+ 6	- 2	- 3	- 2	- 1	- 1
18 ^h	- 4	+22	- 4	-12	0	+ 5	- 2	- 3
19 ^h	- 9	+ 7	-11	+ 2	0	+ 1	- 6	0
20 ^h	-12	- 4	+15	-14	+ 4	+ 3	+ 2	0
21 ^h	- 4	+ 3	+ 4	- 7	+ 6	+ 4	- 3	+ 9
22 ^h	+ 3	- 7	+ 6	- 2	+ 9	- 6	-10	+15
23 ^h	+ 2	-11	-20	+28	+ 6	+13	+ 4	-11
Mean	- 4	+ 2	- 2	- 1	+ 4	+ 3	- 3	+ 2
18 ^h	-12	+ 9	0	-15	- 3	+ 4	-11	+ 1
19 ^h	-11	-25	+ 9	+ 3	- 4	+ 2	+ 5	+ 2
20 ^h	+ 2	+ 6	+18	0	-13	+ 6	+ 4	- 7
21 ^h	- 9	-10	+ 5	+20	+ 4	- 3	- 3	- 9
22 ^h	- 1	+10	+ 9	+ 2	+ 4	+ 2	+ 5	- 8
23 ^h	- 4	+22	0	+ 2	+ 9	+ 3	- 6	- 9
Mean	- 6	+ 2	+ 7	+ 2	0	+ 2	- 1	- 5

10. The sum of the x residuals without regard to sign is 848, and of the y residuals 511, giving a ratio of 1.66 for the probable

errors. The sum of the mean of group residuals is similarly 55 for x and 33 for y , giving a ratio of 1.67. Thus the y observations are more accurate. Since either coordinate would equally well reveal any optical distortion or tilt, we may rest satisfied that these errors are at least as small as is indicated by the y residuals, which are less affected by accidental errors. It may be well to repeat the means for all the plates for y in the more familiar unit of seconds of arc, as below.

+ 0''03	- 0''06	+ 0''03	+ 0''06
- 0''09	- 0''06	- 0''03	- 0''03
+ 0''12	+ 0''09	- 0''09	+ 0''06
0''00	+ 0''06	- 0''03	- 0''15

11. I think it will be admitted that measures capable of giving such means are sensibly free from systematic errors, whatever the accidental errors may be. It should perhaps be repeated here that the measures are made with scales divided only to 3''0, and reading by estimation to 0''.3, so that the largest of the above residuals, 0''.15, is only half the unit of measurement.

The x residuals, though rougher, never reach the unit of measurement.

Two plates with different centres. §§ 12-21.

12. When two plates are taken with centres at the same point of the heavens (the "centre" meaning in this connection the foot of the normal from the centre of the object-glass on the plate), then coordinates on one are connected by simple linear relations with coordinates on the other, whatever may be the differences of scale orientation, refraction, aberration, &c. But for plates with different centres the relations are no longer linear, though, since the centres are not far apart whenever stars are common to both plates, the non-linear terms are small. If $(x_1 y_1)$ are the coordinates of a star on one plate, and $(x_2 y_2)$ those of the same star on the other, and if, further, $(X Y)$ be the coordinates of the centre of plate 2 on plate 1, then

$$x_2 + (Xx_1 + Yy_1)\mu^2, \text{ and } y_2 + (Xx_1/2 + Yy_1/2)\mu^2,$$

where μ is the circular measure of $5'$, are linear functions of x_1 and y_1 . A proof of this proposition has already been given; the following is another which represents the relations from a slightly different point of view.

13. Let $(l_1 m_1 n_1)$, $(l_2 m_2 n_2)$ be direction-cosines of the same star referred to two different sets of axes. Then the ratios $\left(\frac{\mu l}{n}, \frac{\mu m}{n}\right)$ are in each case what we have called "standard

coordinates" referred to the extremity of the n axis as plate-centre. Further, we know that there are linear relations,

$$\left. \begin{aligned} l_2 &= a_1 l_1 + a_2 m_1 + a_3 n_1 \\ m_2 &= b_1 l_1 + b_2 m_1 + b_3 n_1 \\ n_2 &= c_1 l_1 + c_2 m_1 + c_3 n_1 \end{aligned} \right\} \dots \dots \dots (5)$$

Hence, if we put $\mu\xi = \frac{l}{n}$, $\mu\eta = \frac{m}{n}$, we have

$$\mu\xi_2 = \frac{l_2}{n_2} = \frac{\mu a_1 \xi_1 + \mu a_2 \eta_1 + a_3}{\mu c_1 \xi_1 + \mu c_2 \eta_1 + c_3}, \quad \mu\eta_2 = \frac{\mu b_1 \xi_1 + \mu b_2 \eta_1 + b_3}{\mu c_1 \xi_1 + \mu c_2 \eta_1 + c_3} \dots \dots (6)$$

The coordinates (X, Y) of the centre of the second plate on the first are given by putting $\xi_2 = 0$, $\eta_2 = 0$.

Hence

$$\left. \begin{aligned} \mu a_1 X + \mu a_2 Y + a_3 &= 0 \\ \mu b_1 X + \mu b_2 Y + b_3 &= 0 \end{aligned} \right\} \dots \dots \dots (7)$$

and thus

$$\mu X = \frac{a_2 b_3 - a_3 b_2}{a_1 b_2 - a_2 b_1} = \frac{c_1}{c_3}, \quad \mu Y = \frac{c_2}{c_3} \dots \dots \dots (8)$$

by the properties of direction-cosines.

Thus the denominator in the expressions for ξ_2 and η_2 becomes

$$c_3(1 + \mu^2 X \xi_1 + \mu^2 Y \eta_1),$$

and we have, multiplying up by the factor in brackets,

$$\left. \begin{aligned} \xi_2 + \mu^2(X \xi_1 \xi_2 + Y \eta_1 \xi_2) &= \frac{a_1}{c_3} \xi_1 + \frac{a_2}{c_3} \eta_1 + \frac{1}{\mu} \cdot \frac{a_3}{c_3} \\ \eta_2 + \mu^2(X \xi_1 \eta_2 + Y \eta_1 \eta_2) &= \frac{b_1}{c_3} \xi_1 + \frac{b_2}{c_3} \eta_2 + \frac{1}{\mu} \cdot \frac{b_3}{c_3} \end{aligned} \right\} \dots \dots (9)$$

14. Thus, to correct ξ_2 and η_2 so that they may be linear functions of ξ_1 and η_1 we must add to each its product by

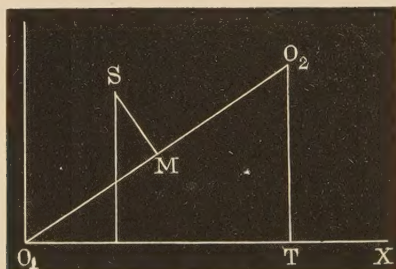


FIG. 1.

$\mu^2(X \xi_1 + Y \eta_1)$. The geometrical meaning of this latter quantity is as follows.—Let O_1 be the centre of the first plate, O_1X the

axis of x ; O_2 the centre of second plate as projected on the first; and S a star. Then

$$\left. \begin{aligned} X\xi_1 + Y\eta_1 &= O_1S \times O_1O_2 \cos SO_1O_2 \\ &= O_1O_2 \times O_1M \end{aligned} \right\} \dots \dots (10)$$

where SM is the perpendicular from S on O_1O_2 .

The correction is thus directly proportional to (a) the distance between centres of plates; (b) to the abscissæ of stars parallel to the line joining centres; and (c) to the coordinate to be corrected.

15. In the present investigation, in which plates of constant size ($2^\circ \times 2^\circ$ approximately) are compared with a large plate which includes them completely, the range of values of this correction varies directly as a , the distance between centres; for of the other two factors, b has nearly the same range, and c exactly the same range in all cases.

16. When the plate centres are near together, as they are in practically dealing with photographs, and the axis of $(\xi_1 \eta_1)$ nearly in the same direction as those of $(\xi_2 \eta_2)$, we may write the corrections

$$X\xi^2 + Y\xi\eta, \quad X\xi\eta + Y\eta^2,$$

where $(\xi \eta)$ denote either $(\xi_1 \eta_1)$ or $(\xi_2 \eta_2)$, as is most convenient. For $\xi_2 \eta_2$ differ from $\xi_1 \eta_1$ by expressions of the form $a\xi + b\eta + c$, of which the c is the most important part, a and b being small; and thus

$$(X\xi_1\xi_2 + Y\xi_2\eta_1) \text{ differs from } (X\xi_1^2 + Y\xi_1\eta_1)$$

by an expression the important part of which is linear, and thus included in the subsequent linear solution, the small terms being negligible.

17. Further (X, Y) may be regarded either as the coordinates of plate-centre 2 on plate 1, or as the reversed coordinates of plate-centre 1 on plate 2. This is easily seen by the following expressions for the constants $a_1 a_2 a_3 b_1 b_2$, &c.; which are all expressible in terms of three of them. Let these three be $a_2=r$, $a_3=q$, $b_3=p$, all small quantities (since the axes are nearly coincident), and let us neglect products of p, q, r . Then $a_1^2 = 1 - q^2 - r^2 = 1$ to first order; and since

$$c_1 b_1 + a_2 b_2 + a_3 b_3 = 0 = b_1 + r b_2,$$

and

$$b_1^2 + b_2^2 = 1 - b_3^2 = 1$$

we have

$$(1 + r^2)b_2^2 = 1, \text{ or } b_2 = 1$$

and thus

$$b_1 = -r. \quad \dots \dots (11)$$

Similarly,

$$c_1 = -q, \quad c_2 = -p \quad \dots \dots (12)$$

so that the scheme of coefficients is

$$\begin{array}{ccc} \mathbf{I} & r & q \\ -r & \mathbf{I} & p \\ -q & -p & \mathbf{I} \end{array}$$

[It is easy to deduce the scheme to the second order, obtaining

$$\begin{array}{lll} \mathbf{I} - \frac{1}{2}(q^2 + r^2), & r - \frac{1}{2}pq, & q - \frac{1}{2}pr \\ -r - \frac{1}{2}pq, & \mathbf{I} - \frac{1}{2}(p^2 + r^2), & p - \frac{1}{2}qr \\ -q - \frac{1}{2}pr, & -p - \frac{1}{2}qr, & \mathbf{I} - \frac{1}{2}(p^2 + q^2) \end{array}$$

but we shall not want more than the first order.]

Thus we have

$$\xi_2 = \frac{\xi_1 + r\eta_1 + q}{1 - q\xi_1 - p\eta_1} \quad \eta_2 = \frac{\eta_1 - r\xi_1 + p}{1 - q\xi_1 - p\eta_1} \quad \dots \quad (13)$$

and thus we see that if $-q=X$, $-p=Y$, the coordinates of centre 2 on plate 1, then to the first order (q, p) or $(-X, -Y)$ are the coordinates of centre 1 on plate 2, viz. the values of $\xi_2\eta_2$ when $\xi_1=0$, $\eta_1=0$.

18. The above method of correcting $\xi_2\eta_2$ so that they may be linear functions of $\xi_1\eta_1$ also holds good when these coordinates are not standard coordinates but actual measures, affected with errors of orientation, scale value, refraction aberration, &c. To make the correction in its simplest form, however, it is advisable first to transform one set of coordinates so that they may be *nearly* the same as the other; so that in the small corrections

$$Xx_1x_2 + Yy_1x_2 \text{ and } Xx_1y_2 + Yy_1y_2$$

we may put $x_1=x_2$ and $y_1=y_2$ and obtain

$$Xx^2 + Yxy, \quad Xxy + Yy^2$$

without sensible error. We need not, however, trouble to eliminate any constant difference between x_1 and x_2 or y_1 and y_2 ; for if

$$x_1 = x_2 + c \quad y_1 = y_2 + f$$

with sensible accuracy for substitution in these small terms, then

$$Xx_1x_2 + Yy_1x_2 = (Xx_2^2 + Yx_2y_2) + (Xc + Yf)x_2,$$

and the term $(Xc + Yf)x_2$ will be absorbed in the linear expression subsequently determined.

19. Thus to compare measures x_1y_1 on a plate with measures x_2y_2 in nearly the same directions on another plate whose centre has coordinates XY on the first; correct x_2y_2 by adding the expressions

$$\mu^2[X(x_2+c)^2 + Y(x_2+c)(y_2+f)], \quad \mu^2[X(x_2+c)(y_2+f) + Y(y_2+f)^2]$$

where c and f are any constants, conveniently chosen so as to make the average corrections small; the corrected x_2y_2 will then be linear functions of x_1 and y_1 , if there is no optical distortion on either plate. Conversely if after such correction the x_2y_2 are found to be linear functions of the x_1 and y_1 , then there is no optical distortion; and any residuals that cannot be satisfied by linear expressions indicate optical distortion.

20. In dealing with plates taken by an instrument for which we have no direct measures of the position of the centre, *i.e.* of the foot of the normal from centre of object-glass on the plate, we must be prepared for errors in the estimated position of this centre, *i.e.* in X and Y . If we use erroneous values of X and Y , then the residuals will remain affected by expressions of the same form as those considered above, *viz.*

$$X'x^2 + Y'xy \text{ and } X'xy + Y'y^2$$

where X' and Y' are the errors in X and Y . We thus have an obvious test of the correctness of the position of the plate-centre from the measures themselves, distinguishable from other sources of error by the form of these expressions.

21. One convenient practical method of applying these corrections has been indicated in *Monthly Notices*, vol. lv. pp. 108, 109, for the case when X and Y are equal. If $Y = mX$, then

$$Xx^2 + Yxy = X(x + \frac{1}{2}my)^2 - \frac{1}{4}Xm^2y^2,$$

and when m is 0.5 (or less), the second term may be neglected in the case considered, *viz.* two plates overlapping one square degree at a corner of each. In any case the second term Xm^2y^2 can be quickly applied as another correction, either by a diagram or a table. The advantage of throwing the expression into this form is that the first term can be applied by a diagram of straight lines, *viz.* lines parallel to

$$x + \frac{1}{2}my = 0.$$

Since $Xx^2 + Yxy = \text{const.}$ represents a hyperbola, it can be represented in the form

$$ax'^2 - by'^2 = \text{const.}$$

in an infinite number of ways (x' and y' being linear functions of x and y), of which

$$X(x + \frac{1}{2}my)^2 - \frac{1}{4}Xm^2y^2$$

is one; and thus the correction can be applied by a pair of straight-line diagrams in an infinite number of ways. Which of these should be chosen is a matter of practical convenience. The particular form given above seems as convenient as any.

In the present paper these corrections are not applied to the individual stars, but to means of groups only, and no special practical device is necessary.

Comparison of Arequipa and Oxford Plates.

22. We now proceed to the comparison of an Arequipa plate taken with the Bruce photographic doublet with an exposure of 10^m on 1896 August 17, centre at $22^h 50^m + 27^\circ 5'$; with two Oxford plates. The relation of the plates is shown by the diagram (fig. 2).

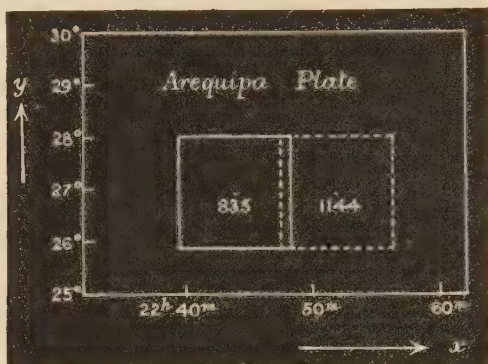


FIG. 2.

23. It was found that the coordinates of the centre of Arequipa plate on plate 1144 are approximately

$$-12.3 \text{ and } +7.0$$

in *réseau* intervals of twelve to the degree, and hence the coordinates $x_2 y_2$ of the Harvard plate when corrected by '000001 multiplied by

$$-26x^2 + 15xy \text{ and } -26xy + 15y^2$$

ought to be linear functions of $x_1 y_1$, the measures of the same stars on 1144.

24. Similarly the coordinates of the centre of Arequipa plate on 835 are approximately

$$+9.2 \text{ and } +6,$$

and thus the corrections are '000001 multiplied by

$$+19x^2 + 13xy \text{ and } +19xy + 13y^2.$$

25. For comparison of the Arequipa plate with these two Oxford plates, copies were made of the Harvard positive on plates the same size as the Oxford plates, and adjusted to fit as exactly as possible the same region.

Plate 1144 (A) and its corresponding copy (B).

26. A preliminary comparison of measures on A and B showed that it was necessary to apply to the measures of A the corrections

$$+ \cdot 173 - \cdot 010x + \cdot 002y \text{ to } x$$

and

$$+ \cdot 130 - \cdot 002x - \cdot 010y \text{ to } y,$$

in order to get measures near those of B. This was done at once to all the measures of A, so that the corresponding stars were easily identified on B, 291 stars in all. These were divided into groups according to the position on the plate, each coordinate being divided into four, viz. : 0·0 to 6·0, 6·0 to 13·0, 13·0 to 20·0, 20·0 to 26·0

The mean coordinates of the centres of the groups *referred to the centre of plate 1144* were thus approximately (the unit being 5' or one *réseau* interval)

$x = -10\cdot0$	$- 3\cdot5$	$+ 3\cdot5$	$+ 10\cdot0$
$y = +10\cdot0$	$+ 10\cdot0$	$+ 10\cdot0$	$+ 10\cdot0$
$x = -10\cdot0$	$- 3\cdot5$	$+ 3\cdot5$	$+ 10\cdot0$
$y = + 3\cdot5$	$+ 3\cdot5$	$+ 3\cdot5$	$+ 3\cdot5$
$x = -10\cdot0$	$- 3\cdot5$	$+ 3\cdot5$	$+ 10\cdot0$
$y = - 3\cdot5$	$- 3\cdot5$	$- 3\cdot5$	$- 3\cdot5$
$x = -10\cdot0$	$- 3\cdot5$	$+ 3\cdot5$	$+ 10\cdot0$
$y = -10\cdot0$	$-10\cdot0$	$-10\cdot0$	$-10\cdot0$

(J)

The distances between centres of groups are nearly, though not quite, equal ; and as represented in the table, they correspond diagrammatically with their positions on plate 1144 ; and this arrangement will be adopted for the residuals in what follows without further explanation.

27. The following are the mean residuals for the groups expressed in units of '0001 of a *réseau* interval or 0''·03, taken in the sense Arequipa minus Oxford.

In x .				In y .			
+ 6	+ 4	+ 15	+ 61	+ 4	+ 10	+ 8	+ 12
+ 16	+ 8	+ 13	+ 52	+ 49	+ 36	+ 7	- 24
+ 13	+ 11	+ 12	+ 45	+ 72	+ 21	- 19	- 76
+ 15	+ 13	+ 20	+ 52	+ 81	+ 28	- 22	- 131

(K)

28. From these we first remove any outstanding linear terms. The x residuals require further correction by $-\cdot 00017x$, $-\cdot 0022$, and the y residuals by $+\cdot 00051x - \cdot 00014y - \cdot 0003$. These coefficients are obtained in the following manner :—

Add together the first two columns and subtract the second two: divide by $8 \times (10 + 35) = 108.0$; the result is the coefficient of x .

Similarly lowest two rows minus highest two rows gives coefficient of y .

Finally mean of all residuals gives the constant term.

This process was uniformly adopted in other similar cases which follow.

After applying these corrections the residuals become

In x .				In y .			
+ 1	- 12	- 13	+ 22	- 64	- 25	+ 9	+ 46
+ 11	- 8	- 15	+ 13	- 10	+ 10	+ 17	+ 19
+ 8	- 5	- 16	+ 6	+ 23	+ 5	+ 1	- 23
+ 10	- 3	- 8	+ 13	+ 41	+ 21	+ 7	- 73

(L)

29. To these we must apply the terms specified in § 23, representing the distance between centres of plates, viz. (omitting a factor 000001) :—

To x , $-26x^2 + 15xy$, and to y , $-26xy + 15y^2$.

[But we may be prepared for an error in the estimated centre of either plate, though the discussion in §§ 3-11 seems to show that the Oxford plates are fairly centred.] In applying such a term as $-26x^2$, which has values $-26, -3, -3, -26$, for the columns of the table, it is well to add the constant $+15$ to all the residuals so as to keep the mean value small. Similarly in applying $+15y^2$ we add the constant -8 to all. The residuals then become

In x .				In y .			
- 25	- 5	+ 4	+ 26	- 31	- 9	+ 7	+ 27
- 5	+ 2	- 1	+ 7	- 7	+ 7	+ 8	+ 4
+ 2	+ 9	- 6	- 10	+ 8	- 4	- 2	- 20
+ 14	+ 14	- 1	- 13	+ 22	+ 18	+ 22	- 40

(M)

30. It is clear that there is still something systematic about the residuals, which an error of centring will not explain. Thus the y residuals require further correction by a term in xy about as large as that already applied, viz $-26xy$; but if this is due to an error in the X coordinate of one of the centres, it implies the existence of a correction to the x residuals of $-26x^2$, which would decrease the first and fourth columns compared with the second and third. But the second and third columns are already slightly greater than the other two, and this further correction would upset the x residuals.

31. Let us examine what expressions of the form $Ax^2 + Bxy + Cy^2$, and $Dx^2 + Exy + Fy^2$ would satisfy the x and y residuals

respectively. The values of x being $-10, -3.5, +3.5$ and $+10$, those of x^2 are $100, 12, 12$ and 100 ; and thus in the x residuals first column + fourth column - (second column + third column) is affected by $4(200-24)A=704A$, whereas the terms Bxy and Cy^2 are eliminated. Similarly the coefficients C, D , and F may be obtained independently of the others.

For the coefficients B and E we must add the four terms in the right-hand top corner to those in the left-hand bottom corner, and subtract the other two corners. The result will be affected by $4(10 \times 10 + 10 \times 3.5 + 10 \times 3.5 + 3.5 \times 3.5)B$, or $729B$.

Obtaining the values of the coefficients in this way, we thus deduce the following expressions for correction of the residuals (M):—

$$\text{For } x, + 3x^2 - 19xy - 2y^2 = Ax^2 + Bxy + Cy^2,$$

$$\text{For } y, + 11x^2 - 23xy - 3y^2 = Dx^2 + Exy + Fy^2,$$

supplying the factor .000001 throughout. These corrections reduce the residuals to the following:—

In x .				In y .				(N)
-6	-2	-5	+7	-5	-8	-8	+7	
+3	+3	-4	+3	+7	+6	+1	+2	
-2	+6	-5	-2	+6	-11	-3	-6	
-5	+5	+2	+6	+2	+3	+23	-14	

the only noteworthy discrepancy being in the right-hand bottom corner of the y residuals, which is farthest from the centre of the Arequipa plate.

32. If the corrections of the last paragraph were due to imperfect centring we should have $A=E, B=F, D=C=0$; but these conditions are not fulfilled, and hence we must look for some other cause of error. We may remark that the chief points to be explained are:—

- (a) The large value of E compared with A .
- (b) The large value of B compared with F .
- (c) The sensible value of D .

If there is an error of centring, we must add the same constant to A and E ; thus, instead of $+2$ and -26 , their values will be $-2X+2$ and $-2X-26$. If X lies between $+1$ and -13 , A and E are of opposite signs. If $X > +1$, A and E are both negative and $E > A$. If $X < -13$, A and E are both positive and $A > E$. In this last case the X coordinate of the foot of the normal from centre of OG on plate hitherto called the centre, instead of being -12.3 as in § 23, is -25 , or even greater. But an error of twelve *réseau* intervals or more is unlikely, though possible.

33. The meaning of the terms Bxy and Exy , or numerically $-19xy$ and $-23xy$, can be realised on inspection of the residuals

(M). We see that the NE and SW corners (using the analogy of a map) are positive, and the NW and SE are negative. Moreover, this feature is exhibited in a minor degree by every group of four residuals. Take, for instance, the NW group of the x residuals

$$\begin{array}{r} -25 \quad -5 \\ -5 \quad +2 \end{array}$$

and subtract the NW and SE figures from the NE and SW, we get $-5-5+25-2=+13$.

Let the coordinates of these four points be

$$\begin{array}{ll} x, y + \beta, & x + \alpha, y + \beta, \\ x, y & x + \alpha, y. \end{array}$$

Then if the error at (x, y) be

$$c + ax + by + Ax^2 + Bxy + Cy^2,$$

and we subtract the NW and SE errors from the NE and SW, all the other terms except the Bxy term cut out, and the result is $Ba\beta$.

In the case of the group given above, a is 6.5 and β is 6.5; so that $a\beta=42$. For other groups $a\beta$ is 6.5×7.0 or 7.0×7.0 —not quite uniform in value but nearly so, and dividing by the proper value of $a\beta$, we may represent the values of B and E in different parts of the plate as follows :—

In x .			In y .		
-40	-27	-33	-36	-33	-57
0	-24	-27	-58	+ 2	- 31
-17	0	-19	+ 19	+ 4	-105

Thus the values of B and E are not quite uniform over the plate : the SW corner seems to be different from the rest.

Theoretical formulæ for optical distortion.

34. In proceeding to consider optical distortion, we shall (in the first instance at least) consider it a radial displacement, symmetrical round a centre. This centre will probably be near the geometrical centre of the plate, but is not necessarily coincident with it, nor with the foot of the normal from centre of OG on plate. In what follows we shall speak of the geometrical centre of the plate as "the centre"; of the foot of the normal from centre of OG on plate, to which we have attached the coordinates (X, Y) , as "the normal"; and to the centre of optical distortion as the "optical centre." These three points should be coincident for good adjustment of the instrument, but may not be.

35. Since the distortion does not become infinite at the optical centre, an appropriate expression for the displacement along a radius is

$$f(r) = A_0 + A_1 r + A_2 r^2 + A_3 r^3 + \dots \quad (14)$$

Clearly $A_0 = 0$; and we may also omit $A_1 r$, which means a simple change of scale. Thus

$$f(r) = A_2 r^2 + A_3 r^3 + \dots \quad (15)$$

The displacement parallel to the axes of x and y are thus

$$f(r) \cdot x/r \text{ and } f(r) \cdot y/r,$$

or

$$x (A_2 r + A_3 r^2 + \dots), \quad y (A_2 r + A_3 r^2 + \dots),$$

representative terms being

$$\mu x r^n \text{ and } \mu y r^n$$

where n is some positive integer. Now consider the displacement in x , say $\phi(x, y)$. In the neighbourhood of a point (ξ, η) let the coordinates of a point be $(\xi + x, \eta + y)$.

Then

$$\begin{aligned} \phi(\xi + x, \eta + y) &= \phi(\xi, \eta) + x\phi_\xi + y\phi_\eta \\ &\quad + \frac{1}{2}(x^2\phi_{\xi\xi} + 2xy\phi_{\xi\eta} + y^2\phi_{\eta\eta}) \\ &\quad + \text{higher powers of } x \text{ and } y. \end{aligned}$$

But by our method of dealing with the plate we have eliminated the linear terms

$$\phi + x\phi_\xi + y\phi_\eta$$

whatever they may be. Further, we have found by experiment an expression of the second order, which approximately satisfies the residuals, and this must be identical with the expression

$$\frac{1}{2}(x^2\phi_{\xi\xi} + 2xy\phi_{\xi\eta} + y^2\phi_{\eta\eta})$$

so far as it goes. For the point (ξ, η) we take the centre of plate 1144, and for (x, y) the coordinates of a star referred to this centre. Now

$$\frac{d}{dx}(x r^n) = r^n + n x^2 \cdot r^{n-2}, \quad \frac{d}{dy}(x r^n) = n x y r^{n-2}$$

$$\frac{d^2}{dx^2}(r^n x) = n x r^{n-4} [3r^2 + (n-2)x^2], \quad \frac{d^2}{dx dy}(r^n x) = n y r^{n-4} [r^2 + (n-2)x^2].$$

36. Hence, if we attempted to represent the displacements

M M

in x and y in the neighbourhood of a point whose coordinates are $(\xi\eta)$ by expressions of the form

$$Ax^2 + Bxy + Cy^2$$

$$Dx^2 + Exy + Fy^2,$$

we should find

$$\begin{aligned} A &= 3\xi \cdot \mu n \rho^{n-4} [\rho^2 + \frac{1}{3}(n-2)\xi^2] & D &= \eta \cdot \mu n \rho^{n-4} [\rho^2 + (n-2)\xi^2] \\ B &= 2\eta \cdot \mu n \rho^{n-4} [\rho^2 + (n-2)\xi^2] & E &= 2\xi \cdot \mu n \rho^{n-4} [\rho^2 + (n-2)\eta^2] \\ C &= \xi \cdot \mu n \rho^{n-4} [\rho^2 + (n-2)\eta^2] & F &= 3\eta \cdot \mu n \rho^{n-4} [\rho^2 + \frac{1}{3}(n-2)\eta^2]. \end{aligned}$$

Thus whatever be the value of n we have

$$B = 2D, E = 2C.$$

If $n=2$, the coefficients have the values

$$\begin{aligned} A &= 3k\xi & B &= 2l\eta & C &= k\xi \\ D &= l\eta & E &= 2k\xi & F &= 3l\eta \end{aligned}$$

and unless n is very large the ratios of the coefficients are not very different from these.

37. Whatever be the value of n , if it is not less than 2, we have A E C all of the same sign, and D B F of the same sign. Further

$$A - E = \xi \mu n \rho^{n-4} [\rho^2 + (n-2)(\xi^2 - 2\eta^2)]$$

$$A - C = \xi \mu n \rho^{n-4} [2\rho^2 + (n-2)(\xi^2 - \eta^2)].$$

Thus A is always numerically greater than C and generally greater than E or 2C. Similarly F is generally numerically greater than B, and always than D.

These relations are of the greatest use in determining whether any given systematic errors are due to optical distortion of any kind.

38. We have made no assumption as to the position of the optical centre, and hence it may be anywhere; or there may be several optical centres, each with its own kind of distortion; and the images may suffer a complex displacement which is the resultant of these different distortions. Still this criterion holds: viz. that if over any region we express the displacements in x and y by terms of the second order, then whenever there is a term $2Pxy$ affecting the x residuals there must be terms $(Rx^2 + Py^2)$ affecting the Y residuals, where R is at least as great as P and probably twice or three times as great, and is of the same sign.

39. Such a fundamental relationship must have a geometrical meaning, and on examining a simple case the meaning is seen to be as follows:—

Let a square $A B C D$ (fig. 3) be so distorted that the displacements of all points in the x direction are represented by an expression $+ B x y$, the displacement in the y direction being for

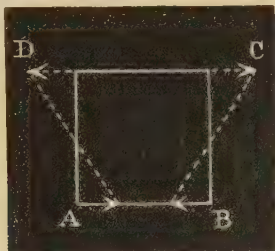


FIG. 3.

the moment not specified. Let the coordinates of the corners be

$$A(-1, -1), B(+1, -1), C(+1, +1), D(-1, +1),$$

so that the displacements of A and C are positive and of B and D negative. The square takes the shape shown in fig. 4.

40. Now let us consider any displacement of points $P Q R S$ by radiation from a centre O , which displaces $P Q R S$ to $p q r s$ (fig. 5). We assume O to lie on the line midway between $P Q$ and $R S$.

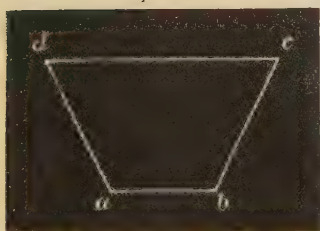


FIG. 4.

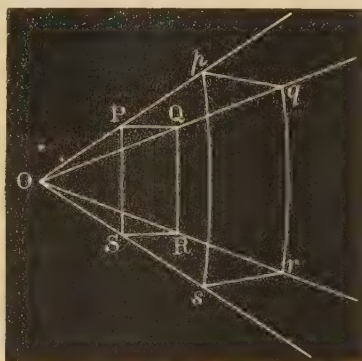


FIG. 5.

Then the lines joining $p s, q r$ will be parallel to $P S, Q R$; but $p q, r s$ will not in general be parallel to $P Q, R S$. For if they were then the ratio $O p : O P$ would be equal to the ratio $O q : O Q$, which represents a simple change of scale of the whole figure. This case is excluded from consideration as involving no distortion; and such change of scale, in either direction, as accom-

panies distortion, is allowed for analytically by linear expressions. Thus we must suppose qr either greater or less than ps . Let it be less as in the figure. Then this means that the ratio Op, OP (in which radii through O are increased) decreases with distance from O , and hence points on PS between P and S , being nearer O , will be displaced proportionately further from O than P and S ; i.e. the line PS will be concave to O , and so will qr .

Hence when we have a pair of sides PQ, RS suffering rotation in opposite directions to pq, rs , the other sides PS, QR become curved.

41. Now returning to figure 3, the sides AD, BC suffer rotation in opposite directions to ad, bc through the term $+Bxy$ in the x coordinate. Hence the sides DC and AB will be curved, which implies a term depending on x^2 in the y coordinate. In this way we can trace the geometrical meaning of the analytical criterion obtained above; though the criterion is perhaps simpler in its analytical form.

42. Returning now to § 31, if we are to explain the values of $A B C D E F$ there given, viz. $+3, -19, -2, +11, -23, -3$, by optical distortion of the kind above considered, then we notice that since $C = -2, E = 2C$ (accurately), and $A = 3C$ (approximately), must be small. The value -2 for C is not necessarily exact, but if we make E and A large we shall introduce a troublesome term in C . We have thus to satisfy as best we can the conditions

$$-2X + 3 = A = 3C, \quad -2X - 23 = E = 2C, \quad C = -2.$$

Solving the equations

$$2X + 3C = 3, \quad 2X + 2C = -23, \quad C = -2$$

by least squares we get $C = +7, X = -14$.

43. Similarly solving

$$2Y + 3D = -3, \quad 2Y + 2D = -19, \quad D = +11$$

by least squares we get $D = +13, Y = -21$; and the theoretical expressions representing optical distortion and error of normal are

$$\text{in } x = 7x^2 - 16xy + 7y^2,$$

$$\text{in } y = 13x^2 - 14xy - 3y^2.$$

Comparing these with the expressions deduced from the residuals in § 30, viz.

$$+ 3x^2 - 19xy - 2y^2$$

$$+ 11x^2 - 23xy - 3y^2$$

the differences (observed—calculated) are

$$+ 10x^2 - 32y - 9y^2$$

$$+ 2x^2 - 9xy + 0y^2,$$

so that even with the assumption of a considerable error of centring ($X = -14$, $Y = -21$) in one or other of the plates compared, and the best value we can get of the optical distortion coefficients, we cannot satisfactorily explain the residuals. The conviction begins to form that the residuals M of § 29 are *not* due to optical distortion, but to some other source of error.

44. But before dismissing entirely the idea of optical distortion let us take another plate, and see whether the suggestions of this plate are borne out by measures in a different part of the Arequipa plate. The suggestions are practically the following:—

(1) That the "normal" is not at the centre of one of the plates, but at a point differing X and Y by $(-14, -21)$. The normal is thus at a distance of nearly $2\frac{1}{2}^\circ$ from the "centre" of the plate—a rather extravagant supposition.

(2) There is optical distortion about an optical centre lying in the left-hand bottom corner of plate 1144, the coordinates of the optical centre with reference to the centre of plate 1144 being $(-2p, -3p)$, where p is not yet determined.

The numerical expressions representing the average optical distortion over the plate are

$$\begin{aligned} &+ 21x^2 + 22xy + 7y^2 \text{ in } x, \\ &+ 11x^2 + 14xy + 33y^2 \text{ in } y. \end{aligned}$$

We thus proceed to the examination of plate 835.

Plate 835 (Oxford) and its corresponding Copy (Arequipa).

45. A preliminary comparison of measures on the plate showed that it was necessary to apply to the measures of 835 the corrections

$$\begin{aligned} &-0.010x + 0.003y \quad \text{to } x \\ &-0.003x + 0.010y + 0.015 \quad \text{to } y \end{aligned}$$

in order to get measures near those of Arequipa plate. This having been done, groups were formed just as in the case of plate 1144 (see § 26), and the following mean residuals found, the unit being, as before, 0.0001 of a *réseau* interval, or $0''.03$:—

In x .				In y .			
(+105)	+15	+16	-2	(-15)	-34	-43	-54
+39	+32	+13	-4	+19	+2	+9	+16
+46	+55	+31	+8	+50	+53	+57	+78
+31	+70	+71	+53	+37	+76	+96	+117

(P)

The group bracketed depends on two stars only, and the x residuals seem exceptional. In deducing the following constants the number for x has been taken as +55, instead of +105.

46. From these we remove linear terms. The x residuals require further correction by

$$+ \cdot 00014x + \cdot 00019y - \cdot 0033,$$

and the y by

$$- \cdot 00008x + \cdot 00062y - \cdot 0029,$$

and the results are :—

In x .	In y .	
(+77) - 4 + 7 - 2	+26 + 2 - 13 - 29	
- 1 + 1 - 8 - 16	+20 - 2 - 1 + 1	(Q)
- 8 + 10 - 4 - 18	+ 7 + 5 + 3 + 19	
-35 + 13 + 24 + 15	-46 - 12 + 2 + 18	

47. To these we must apply terms for difference of normals of the plates. In the first instance, we shall assume the normal coincident with the centre, as in § 24, and apply the corrections

$$+ 19x^2 + 13xy \text{ and } + 19xy + 13y^2,$$

giving results as follows :—

In x .	In y .	
(+72) - 18 + 3 + 19	+13 + 1 0 - 4	
+ 2 - 8 - 15 - 3	+ 7 - 10 - 5 + 2	(R)
+ 5 + 3 - 15 - 15	+ 8 + 1 - 5 + 6	
-14 + 9 + 10 + 10	-21 + 1 + 1 + 5	

48. If we represent these by expressions of the form $Ax^2 + Bxy + Cy^2$, $Dx^2 + Exy + Fy^2$, we get for the values of the coefficients

$$-8x^2 - 3xy - 12y^2$$

and

$$-5x^2 + 4xy + 1y^2.$$

These are small, as may be expected from inspection of the residuals (R).

49. If the normal is not coincident with the centre of the Harvard plate, and the error is as indicated by plate 1144, see §§ 43, 44, so that we should correct residuals (R) by

$$-28x^2 - 42xy, -28xy - 42y^2,$$

then the residuals will require additional correction by

$$+ 20x^2 + 39xy - 12y^2$$

$$- 5x^2 + 32xy + 43y^2,$$

but in these expressions the relations $A=3C$, $E=2C$, $B=2D$, $F=3D$ are not even approximately fulfilled.

If there is an error of normal in the case of this plate it would seem to be in the contrary direction to that of the other plate, at least in the X coordinate, seeing that $C = -12$, and hence A and E should be also negative so far as optical distortion is concerned. Though D is negative ($= -5$), it is small; and we cannot say anything certainly about the sign of B and F, but they should not be large.

Now the error of normal is the combined error of normal in Arequipa and Oxford plates; and though the Arequipa error is the same in both cases, the Oxford plates may have different errors. But even with this freedom of action it was found, on trial, impossible to make any satisfactory compromise which would explain the systematic errors given above on the hypotheses of error of normals and of optical distortion only.

50. I was thus led to examine other possible sources of error. The explanation of the residuals was ultimately found, as I believe, in the curvature of the Arequipa plate, and consequent errors in copying, as will be explained presently.

The following possible sources of error which were considered and found insufficient to explain the residuals may be briefly mentioned:—

51. *Inaccurate Estimation of Centres of Groups.*—In §§ 26–29 it is mentioned that after the application of the main corrections to the individual star places, subsequent corrections were only applied to the mean residuals of groups; and, further, that in so applying them the mean coordinate of a group extending from 0.0 to 5.9 in x and 13.0 to 20.0 in y was assumed to have coordinates 3.0 and 16.5 respectively. To see whether this summary procedure had affected the residuals the actual mean coordinates of the groups on plate 1144 (which is the more difficult to explain) were formed, and were found to differ from the assumed means (viz. 3.0, 9.5, 16.5, 23.0 in each coordinate) as follows:—

In x .				In y .				(S)
-0.4	-0.5	-0.1	-0.5	+0.3	-0.6	-0.2	-0.2	
-0.1	-0.8	+0.1	+0.2	-0.2	+0.3	+0.2	-0.8	
+0.1	+0.4	+0.5	-0.1	-0.8	-0.9	+0.3	-0.4	
0.0	+1.1	+0.2	+0.2	0.0	+0.2	+0.8	-0.1	

52. In § 28 it is stated that the means of groups were corrected, in the first instance, by the quantities $-.00017x - .0022$ and $+.00051x + .00014y - .0003$. Now, it is clear that the use of approximate values for x and y differing from the true values by quantities such as those in table (S) could not produce errors such as those we have to explain. The largest error in the x coordinate of a group is +1.1, which is to be multiplied by .00051, giving +.0006, while we are concerned with quantities four or five times as large as this.

53. Under the same head may be mentioned the application of a term such as Ax^2 to the middle of a group extending (say) from $(x-y)$ to $(x+y)$. We have applied Ax^2 , whereas the proper correction is

$$\frac{1}{2y} \int_{x-y}^{x+y} Ax^2 dx = \frac{A}{6y} [(x+y)^3 - (x-y)^3] = Ax^2 + \frac{1}{3} Ay^2.$$

The error thus made is $\frac{1}{3} Ay^2$; but if the groups are equal in width this is a constant, and only increases all the residuals equally; and since we have subtracted a constant so as to keep the mean residual zero, we may neglect this correction entirely. The groups are not precisely equal in width, the two inner ones extending over seven *résseau* intervals and the outer ones over six, but the error made is very small.

It may be remarked that a term Bxy can be applied to the mid-point of a group without any error.

54. *Refraction*.—The Arequipa plate was taken at some distance from the zenith, and it becomes necessary to examine the refraction correction, which may involve higher powers of the coordinates than the first. A simple method of calculating the differential refraction has been given in *Monthly Notices*, lvii. p. 133 &c., where it is shown that if (X, Y) be the coordinates of the zenith on the plate, μ the coefficient of refraction, then the refractions in x and y are

$$\Delta x = (x - X)t, \quad \Delta y = (y - Y)t^*$$

where

$$t = -\mu(1 + x^2 + y^2) (1 + Xx + Yy)^{-1}$$

so that to the second power of x and y

$$\begin{aligned} \frac{\Delta x}{\mu} &= X - x(1 + X^2) - yXY \\ &\quad + x^2X(2 + X^2) + xyY(1 + 2X^2) + y^2X(1 + Y^2). \end{aligned}$$

The first term X represents the refraction at the centre of the plate, the terms $x(1 + X^2)$ and yXY are included in the general linear terms $ax + by$. It is with the terms of the second order that we are concerned.

55. To calculate X and Y we have

$$\tan q = \tan \lambda \cos (S - A)$$

$$X = \tan (S - A) \sin q \sec (P - q) \quad Y = \tan (P - q).$$

where

$A = 22^h 47^m_{42}$ the R.A. of plate centre.

$P = 62^\circ 5'$ the N.P.D. of plate centre.

$S = 21^h 36^m$ the sid. time of exposure.

$\lambda = 106^\circ 24'$ colatitude of Arequipa.

* There is a mistake of sign in line 19 of p. 136, but the following line is correct: See also p. 135, line 12.

56. It is clear without any actual calculation that $Y = -\tan 45^\circ = -1$ nearly ; and $X = -0.4$ roughly. Thus the terms of the second order are about

$$\begin{aligned}\text{in } x & -\mu[0.9x^2 + 1.3xy + 0.8y^2] \\ \text{in } y & -\mu[3.0x^2 + 1.2xy + 1.2y^2]\end{aligned}$$

where x and y are expressed in circular measure, and μ is the coefficient of refraction $= 0.2$ of a *résseau* interval. Thus, if $x = 1^\circ = 0.017$, $y = 0.017$, the largest value of the correction to y in *résseau* intervals is

$$0.2 \times 5.8 \times (0.017)^2 = 0.0003,$$

and the correction to x is smaller still.

Thus in the present discussion these corrections are too small to have any sensible influence and may be neglected.

57. This does not mean that the correction for refraction to the second order of x and y can be neglected *over the whole Arequipa plate* ; for in the corners of this plate we may have $x = 0.05$, $y = 0.04$; and thus the correction in y at a corner may be

$$0.2 \times (0.0075 + 0.0024 + 0.0019) = 0.0026.$$

a considerable quantity. When, however, we are dealing with a portion only of this large plate, two degrees square, these corrections can be replaced by linear corrections, which are nearly equivalent.

58. *Size of Star Discs.*—No account has been hitherto taken of the magnitudes of the stars. When photographic star discs become at all elongated, there is often a denser nucleus near one end ; and the question arises what to measure, whether the centre of the ellipse or the nucleus. Since the dense nucleus is the first portion of the image to appear on the plate, and thus for faint stars the only portion to appear, it is the best point to measure if we want results consistent between bright and faint stars. But in the brightest stars the nucleus disappears, the whole image being equally dense. Hence there is something of a dilemma in measuring elongated stars, of which there are many examples both on Arequipa and Oxford plates ; and in the case of unequal distribution of bright stars on a plate, we may get systematic errors. A glance at plate 1144 showed that the brighter stars nearly all lay near one diagonal ; and this seemed to afford a possible clue to the discrepancies under consideration. Accordingly the groups were divided into bright and faint stars and the means compared ; but it was soon seen that, though there were unmistakable differences, the effect of such differences was far too small to explain our discrepancies. If the bright stars were omitted near the one diagonal and retained in other places, very little alteration was made in the means.

Curvature of the Plate.

59. While considering all possible sources of these systematic errors, my attention was called by Mr. Bellamy to the fact that the Arequipa photograph was not on plate glass, and I was led to examine its curvature by putting the edges downwards on a piece of paper and running a pencil along them; then turning the paper through 180° and again ruling along the edge with a pencil. In this way the plate was found to have a considerable curvature in opposite senses for the two pairs of parallel edges. In the E. and W. direction the film is concave; in the N. and S. direction, convex. The depth of the middle of an edge below the ends was in the case of the long south edge as much as 2^{mm} or 3^{mm} ; and in all cases a considerable fraction of 1^{mm} . On bending the plate with the hand, it was found that the greatest curvature could be easily doubled or reduced to zero. Mr. Bellamy remembered that in making the copies on 6 in. $\times$ 6 in. plates, he had used a certain amount of pressure. The smaller plate AB (see fig. 6) was placed film downwards on the Arequipa plate

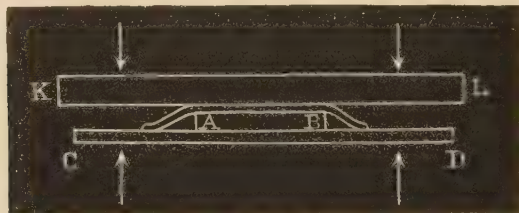


FIG. 6.

CD (film up), then cloth placed at the back of AB, and then a board, KL, on the cloth. Pressure was then applied by Mr. Bellamy's two hands at the points indicated by the arrows. If this pressure had any effect on the curvature of CD it would tend to increase the concavity of the film, which is already concave in the direction CD.

60. Thus we may take it that the Arequipa plate was probably considerably curved when the copy was being made, and it is unlikely that the film of the plate which received the copy was everywhere in contact with the other. Since glass will bend so as to take up the amount of curvature now under consideration, complete contact might have been secured perhaps if special attention had been paid to the point; but it was not, and thus the two films may have had a space between them, as shown in fig. 7.

61. If now the rays of light with which the film of CD was illuminated in order to make the copy were strictly normal to the plate AB, then the image of any point O would fall at *m*.

If, however, the light fell obliquely at an angle θ with the normal, the image of O would fall at l or n , showing a displacement $OmX \tan \theta$ with reference to the former position, while stars at A and B would remain undisplaced. It is not an extravagant supposition to put $\theta = 6^\circ$, even if some care is exercised in holding the plate normal to the incident light; so that $\tan \theta = 0.1$ say. And thus for a displacement of the image such as is in question, amounting to 0.003 of a *réseau* interval or 0.015 of a mm., we only require a separation of the films by

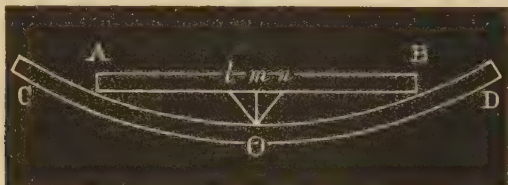


FIG. 7.

0.015 mm., which is quite likely to have occurred in the copying. If we may allow a greater obliquity than 5° for the incident light, a less separation of the films will account for residual errors such as are to be explained.

62. Let z denote the separation of the films at any point (x, y) . Then over any limited area we may put

$$z = c + ax + by + ax^2 + \beta xy + cy^2.$$

If the light is in parallel rays, making $\angle \theta$ with the axis of z , the displacement of the image at xy is in a fixed direction and $= z \tan \theta$. Thus the displacements in x and y are uz and vz , when u and v are constants; and are thus proportional. This affords a simple criterion for judging of this kind of error, viz.

The error in x bears a constant ratio to the error in y .

63. If we have eliminated the linear terms, and are left as before with displacements

$$Ax^2 + Bxy + Cy^2 \text{ in } x \text{ and } Dx^2 + Exy + Fy^2 \text{ in } y,$$

then this criterion gives

$$\frac{A}{D} = \frac{B}{E} = \frac{C}{F}.$$

64. Recurring to § 31, we see at once that this explanation will suit the expressions there given very closely, if we allow for small errors in the coefficients. We there found that after correcting the Arequipa plate for difference of centre from 1144, and for linear terms, the residuals were expressible with some accuracy by

$$\begin{aligned} &+ 3x^2 - 19xy - 2y^2 \text{ in } x \\ &+ 11x^2 - 23xy - 3y^2 \text{ in } y. \end{aligned}$$

Taking 1.2 as the ratio of E to B, and deducing the values of A and C from those of D and F, we should get for A, B, C

$$+9, -19, \text{ and } -3,$$

as compared with

$$+3, -19, \text{ and } -2.$$

65. If we allow for possible error of tilt of the plates we could, of course, satisfy the relations $\frac{A}{D} = \frac{B}{E} = \frac{C}{F}$ exactly, for we have two disposable constants, X and Y, with which to satisfy the equations, which now become

$$\frac{+3+X}{11} = \frac{-19+Y}{-23+X} = \frac{-2}{-3+Y}.$$

Thus we have

$$Y = 19 + (X - 23)(X + 3)/11 = 3 - 22/(X + 3),$$

which leads to the cubic in X,

$$(X + 3)^2 (X - 23) + 176 (X + 3) + 242 = 0,$$

which has a real root between -4 and -5, giving $Y = +22$ or $+24$. The actual errors of centring expressed in *réseau* intervals are half these quantities, owing to the factor $\mu^2 = .00000211$ which occurs in such work (see § 19); and these are quite possible errors of centring, the errors due simply to curvature being expressed by (say)

$$\begin{aligned} & -x^2 + 3xy - 2y^2 \\ & + 11x^2 - 19xy + 19y^2. \end{aligned}$$

But this is treating these expressions rather too seriously.

66. As regards the other plate, No. 835, the residual expressions given in § 48 are

$$\begin{aligned} & -8x^2 - 3xy - 12y^2 \\ & -5x^2 + 4xy + 19y^2. \end{aligned}$$

The ratios A/D, B/E, C/F are not equal, but the quantities A, B, C, D, E, F are all small. If we allow for an error of centring as above, putting

$$\frac{X-8}{-5} = \frac{Y-3}{X+4} = \frac{-12}{Y+1},$$

we get

$$Y = 3 - (X + 4)(X - 8)/5 = -1 + 60/(X - 8).$$

Thus,

$$+ (X + 4)(X + 8)^2 - 20(X + 8) + 300 = 0,$$

or

$$X = -7; Y = 3 - 9 = -6,$$

and the errors due to curvature are

$$-15x^2 - 9xy - 12y^2 \text{ in } x$$

and

$$-5x^2 - 3xy - 5y^2 \text{ in } y.$$

67. The errors in X and Y and the expressions due to curvature of plate do not accord very well with those of the other plate; but there is, after all, no reason why they should. The curvature which the plates concerned had at the time of copying depends on the particular strain to which they were subjected at the time; and as regards errors of centring three plates are involved, exposed under unknown conditions.

68. Further, there is no great necessity for conformity in these expressions all over either plate. Thus local differences, as remarked in § 33, are quite accordant with the present explanation.

69. To sum up, there is no difficulty in accounting for the residual differences between the Oxford plates and the Arequipa plate in this way. They can be accounted for both in character and magnitude. In ignorance of the precise conditions under which these copies were made it is useless to examine more in detail whether the exact shape of the particular plates concerned can be traced in the errors. If the present explanation of the errors is correct, and the curvatures of the plates at the time of exposure, as well as the exact direction of incident light, were known, we ought to be able to establish the causation completely, and this may possibly be done in future experiments. But this opens up a new line of investigation, and must be deferred to a future paper.

Conclusions.

70. The conclusions of the present paper may be thus stated:

- (a) The Oxford astrographic catalogue plates of $130' \times 130'$ show no systematic errors larger than $0'' 15$, which can be attributed to optical distortion.
- (b) A plate taken at Arequipa with a photographic doublet shows no sensible optical distortion over an area covered by two Oxford plates placed side by side, and thus presumably over an area of $4^\circ \times 4^\circ$ at least.
- (c) Certain systematic differences which were found on comparing the Arequipa plate with Oxford plates are not due to optical distortion but to curvature of the plates used in making photographic copies; to the want of complete contact; and to obliqueness in the illumination.
- (d) A simple criterion for the existence of any kind of optical distortion which consists in a radial displacement of stars from a centre, or in the composition of any number of

such displacements, is arrived at in the paper and illustrated.

- (e) A simple criterion is also given for detecting errors in star measures due to curvature of a plate, which is copied on to another without being completely in contact with it, under oblique illumination.

On the Errors of Star Photographs due to optical Distortion of the Object-glass with which the Photograph is taken. Second paper. By H. H. Turner, M.A., F.R.S., Savilian Professor.

1. In a previous paper on this subject I gave the comparison of two Astrographic Catalogue plates taken at Oxford with corresponding portions of a large plate taken at Arequipa, Peru, with a photographic doublet. In each case the coordinates of some 250 stars on the two plates were compared, and the algebraical relations between them were found to be expressible with sensible accuracy by the formulæ

$$x' = c + ax + by + Ax^2 + Bxy + Cy^2$$

$$y' = f + dx + ey + Dx^2 + Exy + Ey^2$$

The linear terms $c + ax + by$ and $f + dx + ey$ were not further considered; but it was pointed out that the values of the coefficients A B C D E F were inconsistent with the idea that these terms were due to optical distortion of the object-glass, if the corresponding displacement of a star image was along the radius from some centre, and a function of the distance from the centre. It was shown to be probable that these terms represented errors of copying the Arequipa plate under oblique illumination; and it was inferred that the optical distortion over a field of at least $4^\circ \times 4^\circ$ was small.

2. In the present paper additional evidence is given of the existence of errors of copying. Measures of a third region still further from the centre of the Arequipa plate are examined in the terms of the second order, which still show no evidence of sensible optical distortion. Finally, the linear terms $c + ax + by$ and $f + dx + ey$, hitherto neglected, are considered; again without finding evidence of optical distortion.

Errors of Copying.

3. In the previous paper no direct evidence was given that the errors were due to copying; but this was shown to be a probable explanation of the discrepancies. It was found that the glass of the Arequipa positive was sensibly curved; and it was supposed that in copying a portion of it on another plate the

films were not in complete contact owing to this curvature. Hence, if the illumination used for copying were slightly oblique, the star images in different parts of the plate would be relatively displaced. Thus, in fig. 1, if $A S B$ be one (curved) film, $A M N B$ the other, in contact with the former at A and B , but not at S ; then if the illumination be in the direction $S M$, the star image S appears at M ; if in the direction $S N$, at N ; while the stars at A and B are the same in both cases.



FIG. 1

4. To test this hypothesis a portion of the Arequipa plate was copied twice on the same plate under illuminations differing considerably in direction, and the distances between the pairs of images measured. The plate was held in the first instance so that light from a gas jet fell at an angle of $+45^\circ$ with the normal, in the plane through the axis of x ; then after a small displacement it was again exposed, with the light making an angle of -45° . It will be remarked that here we are not concerned with the performance of the Bruce doublet in any way. The stars are merely arbitrary points on the plate copied twice over, and the only differences allowable are of the form

$$by + c \text{ in } x \text{ and } -bx + f \text{ in } y$$

representing an arbitrary displacement and rotation in moving the plate slightly, so that the second exposure should not fall on the first. The mean differences of groups in x and y corrected by expressions of the above form were as below, in the notation of the previous paper, the unit being $\cdot 0001$ of a *réseau* interval, or $0''.03$.

In x				In y			
- 5	+ 3	+ 17	- 9	- 21	- 14	- 8	- 7
- 14	+ 7	+ 36	+ 7	- 13	- 9	- 6	- 2
- 13	+ 7	+ 24	+ 6	- 19	- 9	0	+ 8
- 10	- 7	+ 17	+ 8	- 18	- 5	- 1	+ 12

5. In the same way as in the former paper we find that algebraical expressions representing these are :

$$\text{for } x, \text{ const.} + \cdot 00013x + \cdot 00001y - \cdot 000001(19x^2 + 2xy + 6y^2)$$

$$\text{for } y, \text{ const.} + \cdot 00010x - \cdot 00004y - \cdot 000001(1x^2 + 5xy + 2y^2)$$

These expressions reduce the above differences to the following :

In x				In y			
+ 12	- 5	+ 4	- 14	- 3	0	+ 2	+ 1
- 1	- 5	+ 16	- 4	+ 3	+ 2	- 3	- 2
+ 2	- 5	+ 4	- 7	- 1	0	+ 1	+ 2
+ 13	- 11	+ 3	+ 1	+ 2	+ 5	- 3	+ 2

6. If the differences are due to errors of the kind suggested, then both the algebraical expressions and the residuals in x and y should be in a constant ratio. Looking first at the residuals, take the five largest residuals in x , and write under them the corresponding residuals in y .

$$x + 16 + 12 - 14 + 13 - 11$$

$$y - 3 - 3 + 1 + 2 + 5$$

The ratio of the y residuals to the x residuals is thus small, its mean value being $-10/66$; and there is a reason for this, the obliquity of illumination being changed principally in the direction of the x axis, though the position of the plate being only roughly estimated, there was probably a slight change of direction in the other coordinate also. Thus corresponding to the term $-19x^2$ for x , we should expect a term $+3x^2$ in y , whereas we find $-1x^2$; but this need not trouble us, for coefficients smaller than 10 in these terms may be regarded as largely accidental.

In considering the terms of the first order, it must be remembered that we have an unknown rotation of one exposure with reference to the other; thus, instead of

$$13x + 1y \text{ and } 10x - 4y$$

we should write

$$13x + (1+b)y \text{ and } (10-b)x - 4y.$$

Putting $b=13$ we get

$$13x + 14y \text{ and } -3x - 4y,$$

which are approximately in the ratio of 66 to -10 . So that there is no theoretical difficulty in getting this accordance between these terms and the residuals. But this is treating too seriously terms of the magnitude of $-3x - 4y$, which may be in error by several units. It is probably better to put $b=5$ and get

$$+13x + 6y \text{ for } x, \text{ and } 5x - 4y \text{ for } y,$$

of which we can regard as largely accidental the smaller terms, *i.e.* all except the $+13x$.

Hence, the main part of the discrepancy between exposures is represented by

$$\text{Const.} + \cdot 000130 x - \cdot 000019 x^2 \text{ in } x$$

and

$$\text{Zero} \qquad \qquad \text{in } y$$

neglecting small terms as largely accidental.

This can be put into the form

$$\text{Const.} - \cdot 000019 (x - 3\cdot 4)^2$$

and thus $x=3\cdot 4$ is the point, either of closest contact between the two films in copying, or of greatest separation. Since it is more likely that the edges of the superimposed plate were in contact with the film to be copied, with a gap in the middle, rather than that the contact was in the middle with separation at the edges, $x=3\cdot 4$ is probably the point of greatest separation. But this is not material.

7. What *is* rather important is the numerical smallness of the quantity. The obliquity of illumination was purposely exaggerated, and we should have expected at least five times the sort of quantities which were to be explained in the last paper, such as

$$+ 11x^2 - 23xy - 3y^2$$

(see p. 451); whereas, the quantity obtained is only about the same size. This is capable of explanation in more than one way, thus:—

(a) The plates may have been pressed closer together on the present occasion.

(b) The errors dealt with in the last paper may not be wholly due to the copying at Oxford, but partly to the process of copying the original negative at Arequipa.

8. In any case the existence of errors of this kind is fairly well established, and we may now return to the main object of the investigation, the possible existence of optical distortion. The next step was to take a copy of a portion of the Arequipa plate still further from the centre: to compare it with the corresponding Oxford plate as before: to express the residuals, after clearing away linear terms, by expressions of the second order, and to see whether at this greater distance from the centre these were becoming larger than could be explained by errors of copying.

9. The centres of the two plates previously considered occupied the following positions on the Arequipa plate,

No.	x	y	r
1144	+ 12	- 7	14
835	- 9	- 6	11

The new plate taken was with centre,

$$207 \quad -25 \quad +6 \quad 26$$

To the Oxford measures were applied the corrections,

$$+ \cdot 220 - \cdot 010x - \cdot 004y \text{ to } x \\ 0 \quad + \cdot 004x - \cdot 010y \text{ to } y$$

and it was then compared with the corresponding portion of the Arequipa plate. The mean differences were as below, the unit being $0''\cdot 03$.

In x				In y			
+ 22	+ 165	+ 289	+ 368	+ 274	+ 177	+ 70	+ 9
+ 39	+ 151	+ 304	+ 368	+ 183	+ 116	+ 23	- 40
- 14	+ 116	+ 262	+ 311	+ 109	+ 59	- 5	- 35
- 10	+ 96	+ 209	+ 282	+ 37	+ 11	- 13	- 37

10. When corrected by the linear expressions

$$- \cdot 0185 - \cdot 00169x - \cdot 00042y \text{ in } x,$$

and

$$- \cdot 0059 + \cdot 00092x - \cdot 00064y \text{ in } y,$$

these become

In x				In y			
- 36	- 3	+ 3	- 28	+ 59	+ 22	- 21	- 22
+ 8	+ 10	+ 45	- 1	+ 10	+ 3	- 26	- 29
- 15	+ 5	+ 33	- 28	- 20	- 10	- 10	+ 20
+ 16	+ 12	+ 7	- 30	- 50	- 16	+ 24	+ 60

11. Calculating for these corrections of the second order as in the previous paper, we find

$$(+ 32x^2 - 10xy + 16y^2) \times \cdot 000001 \\ (- 9x^2 + 54xy - 17y^2) \times \cdot 000001.$$

Now the centre being at $(-25, +6)$, or the coordinates of the centre of the Arequipa plate on this plate being $(+25, -6)$, the theoretical corrections for difference of normals

$$\mu^2 (Xx^2 + Yxy) \text{ and } \mu^2 (Xxy + Yy^2)$$

(where μ is the circular measure of the *réseau* interval or $\mu^2 = \cdot 0000021$) are respectively

$$+ 53x^2 - 13xy \text{ and } + 53xy - 13y^2,$$

both multiplied by '000001. Hence the outstanding residuals require correction by

$$-21x^2 + 3xy + 16y^2 \text{ for } x$$

and

$$-9x^2 + xy - 4y^2 \text{ for } y.$$

12. Calling these $Ax^2 + Bxy + Cy^2$ and $Dx^2 + Exy + Fy^2$, then if they were due to optical distortion, according to the criterion of the last paper, A, E, C should be of the same sign and in descending order numerically; also F, B, D should be of the same sign and in descending order. This is far from being the case, A and C being conspicuously of opposite signs.

13. On the other hand, if these corrections have their origin in errors of copying we should have

$$\frac{A}{D} = \frac{B}{E} = \frac{C}{F}.$$

These relations are not fulfilled as the numbers stand, but a small "error of normal" could be assigned which would make them fit, *e.g.* if X and Y above instead of being $(-25, +6)$ are really $(-21, +12)$, we should get for the theoretical corrections for difference of normals

$$+44x^2 - 25xy \text{ and } +44xy - 25y^2,$$

and thus

$$Ax^2 + Bxy + Cy^2 = -12x^2 + 15xy + 16y^2$$

$$Dx^2 + Exy + Fy^2 = -9x^2 + 10xy + 8y^2$$

14. Of course this is introducing two new arbitrary constants, and we must therefore not attach too much importance to the result; but it is to be remarked that not even by this device can we make the expressions such as would arise from optical distortion, for we cannot make A of the same sign as C and numerically greater without assuming an extravagant error of normal, viz. 18 *réseau* intervals at least.

15. Hence it would appear from the study of the terms of the second order that not even at this increased distance from the centre of the Arequipa plate is there evidence of sensible optical distortion; such terms of this order as affect the residuals being due to errors of tilt or copying.

16. But, as above stated, my attention was recalled by a remark of Mr. Dyson to the linear terms which may theoretically be larger than those of the second order, with a certain law of optical distortion, and the study of them gave interesting results. In explanation of the fact that I have hitherto neglected them, I may remark that in commencing this rather new investigation my attention was naturally attracted by the newest thing which I encountered. Had no terms of the second order presented themselves, the linear terms would naturally have been carefully

examined. But terms of the second order (not due to a difference of centres) in the comparison of two plates were a new feature. In comparing our astrographic plates the relations are sensibly linear (except for any difference of centring), and in looking for optical distortion on these large plates taken with a doublet, which even near the centre could not be expressed in terms of our plates by linear relations, I felt that at any rate these terms of the second order must be explained, and they seemed to afford the quickest route to the solution of the problem. The time spent in studying them was in no way lost, for I doubt whether I should have come upon the errors of copying by study of the linear terms.

The Linear Terms.

17. Before examining the individual plates the nature of the problem may be briefly stated. Each Oxford plate has been compared with the theoretical sky in the course of the regular work at Oxford, and is therefore as good as a piece of sky, except for accidental errors. Each copy of a portion of the Arequipa plate has been compared with an Oxford plate and thus indirectly with the sky. The formula for reduction of a copy to the sky will include

- (a) The effect of refraction for the Arequipa plate ;
- (b) The effect of aberration for the Arequipa plate ;
- (c) The scale value of the Arequipa plate ;
- (d) An arbitrary orientation and error of centring of the particular copy ;
- (e) The correction for scale of the particular copy due to its distance from the centre of the large plate ;
- (f) the effect of optical distortion.

18. As regards (b) the effect of aberration to the first order is a simple change of scale, and hence it may be considered as included in (c).

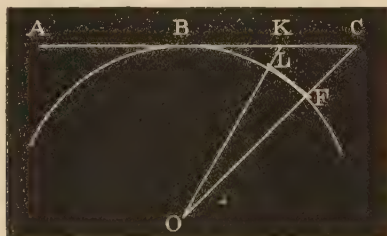


FIG. 2.

The correction (e) requires, perhaps, a little more explanation. Let ABC (fig. 2) be the large Arequipa plate. Then a portion, KC , far from the centre, is compared with an Oxford plate, which may be regarded as touching the celestial sphere at L F ; and

since κc is greater than $L F$, the scale of the Arequipa plate will in this region be relatively larger than at the centre B . The theoretical correction will be given later.

19. Coming to (f') the optical distortion, if there be displacements $\phi(x, y)$ and $\psi(x, y)$ in the coordinates x and y due to optical distortion, then in the neighbourhood of the point (ξ, η) these have the values

$$\phi(\xi, \eta) + x\phi_x + y\phi_y + \frac{1}{2}(x^2\phi_{xx} + 2xy\phi_{xy} + y^2\phi_{yy}) + \&c.,$$

with a similar expression for ψ , and the linear terms are thus

$$x\phi_x + y\phi_y \text{ and } x\psi_x + y\psi_y.$$

As in the previous paper (§ 35), if the displacement be along a radius from some centre, and vary as r^{n+1} , so that the displacements in x and y are $P r^n x = \phi(x, y)$ and $P r^n y = \psi(x, y)$, then

$$\begin{aligned}\phi_x &= P(\rho^n + n\rho^{n-2}\xi^2) & \phi_y &= P n\rho^{n-2}\xi\eta \\ \psi_x &= P n\rho^{n-2}\xi\eta & \psi_y &= P(\rho^n + n\rho^{n-2}\eta^2).\end{aligned}$$

As regards the magnitude of these compared with terms of the second order, we may take as an instance

$$\phi_{xx} = nP\rho^{n-4}\xi\{3\rho^2 + (n-2)\xi^2\}.$$

Hence the ratio of $\frac{1}{2}\phi_{xx}x^2$ to $\phi_x \cdot x$ is

$$\frac{1}{2}n\xi x\{3\rho^2 + (n-2)\xi^2\} \text{ to } \rho^2\{\rho^2 + n\xi^2\}.$$

The limiting cases are $\xi = \rho$ and $\xi = 0$. When $\xi = \rho$ the ratio is

$$\frac{1}{2}nx \text{ to } \rho.$$

When $\xi = 0$ the ratio is zero. Thus if n be large the terms of the second order are more important than the linear terms; but if n be small this is not so.

20. We proceed to the consideration of the three regions measured on the Arequipa plate, and compared with Oxford plates 1144, 835, and 207. In the first instance the Oxford measures, say x and y , were corrected to x' and y' , values near those of the Arequipa copies, to help in readily identifying the proper stars. The values of $x' - x$ and $y' - y$ are given in §§ 26 and 45 of the former paper and § 9 of the present one, and are as below:

Plate.	$x' - x.$	$y' - y.$
1144	$-.010x + .002y$	$-.002x - .010y$
835	$-.010x + .003y$	$-.003x - .010y$
207	$-.010x - .004y$	$+.004x - .010y$

21. We then found that if (X, Y) denote measures on a copy of the Arequipa plate, and (x', y') denote the Oxford measures

corrected as above, the values of $X - x'$ and $Y - y'$ were as follows, omitting a constant :

Plate.	$X - x'$.	$Y - y'$.
1144	+ '00017X + '00000Y	- '00051X + '00014Y
835	- '00014X - '00019Y	+ '00008X - '00062Y
207	+ '00169X + '00042Y	- '00092X + '00064Y

22. Finally, in the course of the regular work at Oxford the following expressions were found for reduction of (x, y) to (ξ, η) , the standard coordinates for stars in the theoretical sky :

Plate.	$x - \xi$.	$y - \eta$.
1144	+ '00750\xi + '01435\eta	- '01418\xi + '00756\eta
835	+ '00835\xi + '00792\eta	- '00769\xi + '00826\eta
207	+ '00781\xi + '00117\eta	- '00150\xi + '00791\eta

23. We thus obtain for $X - \xi$ and $Y - \eta$ the following values:

Plate.	$X - \xi$.	$Y - \eta$.
1144	- '00244\xi + '01624\eta	- '01658\xi - '00242\eta
835	- '00189\xi + '01068\eta	- '01055\xi - '00246\eta
207	- '00058\xi - '00244\eta	+ '00162\xi - '00153\eta

There is thus a well-marked difference in scale value between the copies, but this is in great measure due to the cause specified in (e) of § 17, and we must now estimate this effect numerically.

24. The relations between coordinates (x_1, y_1) on one plate and (x_2, y_2) of the same star on another plate, whose centre is at (XY) on the first, are approximately

$$x_2 = \frac{x_1 - X}{1 + Xx_1 + Yy_1} \quad y_2 = \frac{y_1 - Y}{1 + Xx_1 + Yy_1}$$

or

$$x_2(1 + Xx_1 + Yy_1) = x_1 - X,$$

or substituting on the left

$$x_1 = x_2 + X \text{ and } y_1 = y_2 + Y$$

$$x_2(1 + X^2 + Y^2) + Xx_2^2 + Yx_2y_2 = x_1 - X;$$

and similarly

$$y_2(1 + X^2 + Y^2) + Xx_2y_2 + Yy_2^2 = y_1 - Y.$$

Thus the change of scale is represented by the factor $(1 + X^2 + Y^2)$, and depends simply on the distance between centres, as it should. If X and Y are expressed in *réseau* intervals, we must multiply $X^2 + Y^2$ by '0000021, and the products for the three plates are (see § 9).

Plate.	Product.
1144	'00041
835	'00024
207	'00138

25. In applying these corrections we may at the same time rotate the axes so as to reduce the coefficients of y and x in x and y respectively. If these were exactly equal, and of opposite sign, we could make them both zero by a simple rotation of axis, writing

$$x_2 \text{ for } (1 - \frac{1}{2}\theta^2)x_1 - \theta y_1$$

and

$$y_2 \text{ for } +\theta x_1 + (1 - \frac{1}{2}\theta^2)y_1.$$

But the equality is only approximate, and we choose for θ the numerical mean of the two values, as below :

Plate.	$+\theta$.	$\frac{1}{2}\theta^2$.
1144	-.01641	.00013
835	-.01061	.00005
207	+.00203	.00000

26. The corrected values of $X-\xi$ and $Y-\eta$ thus become

Plate.	$X-\xi$.	$Y-\eta$.
1144	-.00272\xi - .00017\eta	-.00017\xi - .00271\eta
835	-.00228\xi + .00007\eta	+.00006\xi - .00265\eta
207	-.00196\xi - .00041\eta	-.00041\xi - .00291\eta

27. The correction for differential refraction on the Arequipa plate must now be calculated (the refraction and aberration for the Oxford plates are included in the formulæ, reducing them to the theoretical sky). Particulars are given in § 50 of the preceding paper for calculating X_0 , Y_0 , the coordinates of the zenith on the Arequipa plate, which are thus found to be $X_0 = -0.44$ and $Y_0 = -0.99$; and the corrections for differential refraction to the first order, viz.

$$+\mu(1 + X_0^2)X + \mu X_0 Y_0 \cdot Y \text{ in } x$$

and

$$+\mu X_0 Y_0 \cdot X + \mu(1 + Y_0^2)Y \text{ in } y$$

(where μ is the coefficient of refraction and may be taken as $57''$, .00028 in circular measure) thus become

$$+.00033X + .00011Y \text{ and } +.00011X + .00054Y.$$

The effect of refraction is to decrease X and Y by these expressions : and hence the measured X and Y are too small to this extent. To compare X and Y with ξ and η as if there were no refraction we must therefore add the above expressions to X and Y , making $X-\xi$ and $Y-\eta$ larger in consequence, as below.

Plate.	$X-\xi$.	$Y-\eta$.
1144	-.00239\xi - .00006\eta	-.00006\xi - .00217\eta
835	-.00195\xi + .00018\eta	+.00017\xi - .00211\eta
207	-.00163\xi - .00030\eta	-.00030\xi - .00237\eta

28. If there were no optical distortion, then denoting the value of

$$X - \xi \text{ by } a_1\xi + b_1\eta, a_2\xi + b_2\eta, \text{ \&c.}$$

and of

$$Y - \eta \text{ by } +b_1\xi + e_1\eta, +b_2\xi + e_2\eta, \text{ \&c.}$$

we should expect $a_1=a_2=a_3=e_1=e_2=e_3$, and $b_1=b_2=b_3=0$. The departure from this state of things is quite sensible, and it remains to examine whether the departure can be attributed to optical distortion.

29. The values of the coefficients a , b , and e have been given in §19; but the symbols there employed have since been used in rather different senses. Let us put (u, v) for the coordinates of the centre of a plate or region referred to the optical centre, and let $r^2=u^2+v^2$. Then if the distortion is Pr^n , we have the following values for the coefficients a , b , and e :—

$$a = Pr^{n-2} (r^2 + nu^2), \quad b = Pr^{n-2} \cdot uv, \quad e = Pr^{n-2}(r^2 + nv^2).$$

Thus

$$\frac{a-e}{b} = \frac{u^2 - v^2}{uv},$$

a relation which is independent of n , and affords a simple test of the existence of this kind of optical distortion.

30. Assuming that the optical centre is at the centre of the Arequipa plate, the values of u and v for the three plates are given in §9 under the head x and y : and the following tabular statement shows how far the above relation is fulfilled.

Plate.	$a-e$.	b .	$\frac{a-e}{b}$	u^2-v^2	uv	$\frac{u^2-v^2}{uv}$
1144	-00022	-00006	+3.7	+95	-98	-1.0
835	+00016	+00017	+1.0	+45	+54	+0.8
207	+00074	-00030	-2.5	+589	-150	-3.9

There is a considerable discrepancy in the case of Plate 1144; but the coefficient b is small, and may be largely accidental. It may be remembered that this plate is a good deal affected by errors of copying.

31. Since $b/uv = Pr^{n-2}$, a comparison of the three values of b/uv should be some guide to the value of n . Take the second and third plates, which seem fairly accordant. The values of b/uv are in the ratio 31 to 20, which is $(r_2/r_3)^{n-2}$. But $r_2/r_3 = 11/26$. This suggests that $n-2$ is negative, the actual value given by this relation being $n-2 = -0.5$ or $n = 1.5$. If for the same plates we took $(a-e)/(u^2-v^2)$, which should also give Pr^{n-2} , the ratio is 36 to 13, giving $n-2 = -1.2$ or $n = 0.8$. Hence the value $n=1$, implying a law of displacement $=Pr^2$, with displacements Prx and Pry in x and y , does not seem unlikely.

32. Calculating on this supposition

$$\frac{a}{P} = r + \frac{u^2}{r}, \quad \frac{b}{P} = \frac{un}{r}, \quad \frac{e}{P} = r + \frac{v^2}{r}$$

for the three plates, we get the following values :

Plate.	a	b	$\frac{e}{P}$
1144	+ 24	- 6	+ 17
835	+ 18	+ 5	+ 14
207	+ 51	- 6	+ 27

33. It remains to assign the value of P . From consideration of this table in conjunction with that of § 27, we may put $P = +.00002$, and then subtract these theoretical values of a , b , and e from those of § 27. If this law of distortion is satisfactory, we ought then to get the values of a and e all equal, and the values of b all zero. The old values and the new are collected in the following table. To show more clearly the differences of a and e , the mean value -210 has been subtracted from all the old values, and the mean value -261 from all the new values ; the unit is as before, $.00001$.

Plate.	Old Values.			New Values.		
	a	e	b	a	e	b
1144	-29	- 7	- 6	-26	+ 10	+ 6
835	+ 15	- 1	+ 18	+ 30	+ 22	+ 8
207	+ 47	-27	-30	- 4	-30	-18

34. We have not therefore improved things very much. The mean numerical value of b is reduced from 18 to 11, but that of a and e only alters from 21 to 20. Nor can we get any much better result by taking another value of n : it can be shown that there is an inherent difficulty in satisfying the conditions, independent of n . For write the old values of a and e which it is desired to bring into accordance in the order of magnitude, viz. $+47 +15 -1 -7 -27 -29$, then the corresponding expressions $Pr^{n-2}(r^3 + nu^2)$ &c. (see § 29) should fall into approximately the same order. Picking these out, and substituting the values of u , v , and r , from § 9 we get the following series, which should be in order of magnitude.

$$.26^n(1 + .8n), 11^n(1 + .8n), 11^n(1 + .3n), 14^n(1 + .3n), 26^n(1 + 1n), 14^n(1 + .8n)$$

When n is small we may take logarithms to base e , and neglecting powers of n above the first we can divide out by n . Thus the first term is $n(\log. 26 + .8)$, and dividing by n the following numbers should be in order :—

$$1.8 \quad 0.9 \quad 0.4 \quad 0.7 \quad 1.1 \quad 1.1$$

which is not even approximately the case. Large values of n are excluded by other considerations.

35. Hence it is no easier to detect optical distortion in the linear terms than in those of the second order, and we must regard the discrepancies as due to accidental causes—perhaps errors of copying—in the same way as the second order terms.

36. Thus we may feel some confidence that the optical distortion over a considerable field is small; and that accurate positions of stars may be obtained from these large plates if proper precautions be taken. I do not think, however, that the present line of examination is the best way of arriving at precise information about a small optical distortion; the method of trails, mentioned by Captain Hills, R.E., at the April meeting of the Society, seems better.

In another paper I have examined the necessary formulæ of reduction, and find them very simple; and I think the method will prove to be a very easy and direct way of measuring the optical distortion of a lens. The foregoing investigations will, however, serve to show how two plates may be compared when we have no independent information as to the optical distortion.

Conclusions of the Present Paper.

1. The errors of copying due to curvature of plate and oblique illumination, suggested in the last paper, were reproduced by direct experiment. The numerical value found was smaller than was expected, though this is capable of explanation. (§§ 1-7.)

2. A third region, including stars up to 4° from the centre of the Arequipa plate, was compared with an Oxford plate, and the terms of the second order gave no evidence of optical distortion. (§§ 8-15.)

3. The terms of the first order for all three plates dealt with in this paper and the last were examined, and gave no evidence of optical distortion. (§§ 16-35.)

4. Hence it seems possible to obtain good results over a region of (say) $5^\circ \times 5^\circ$ with a photographic doublet.

5. Further experiments are, however, desirable by the method of star-trails, as indicated elsewhere.

On the Curvature of Star-trails on a Photographic Plate as a Means of Investigating Optical Distortion. By H. H. Turner, M.A., F.R.S., Savilian Professor.

1. In the discussion on my paper on optical distortion, read at the April meeting of the Society, Captain Hills, R.E., mentioned a simple method of investigating the distortion on a wide-angle photograph, which he had used in practice—viz. to take a series of star-trails on the plate and to compare their curvature with

the theoretical curvature. If the field be not too large, the theoretical curvature is easily calculated from the formula

$$y = x^2 \times \frac{1}{2} \tan \delta,$$

where x y are expressed in circular measure.

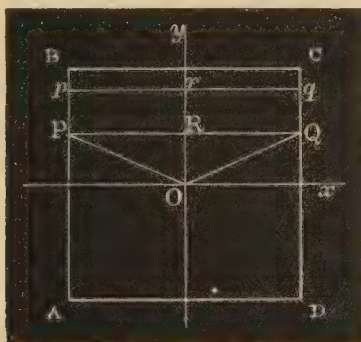
But with large fields it becomes necessary to employ a more accurate formula and to correct for refraction. If spherical coordinates are employed such formulæ are troublesome, but with rectangular coordinates they can be put into a simple form.

2. Let us consider first how optical distortion would affect the curvature of trails in different parts of the plate. We shall suppose the distortion to be a displacement of points along the radius from the centre of the field, and varying as some power of the radius. To fix the ideas, let us suppose it varies as the cube of the radius, so that

$$\Delta r = Ar^3,$$

and thus

$$\Delta x = Ar^2x, \quad \Delta y = Ar^2y.$$



Let A B C D be a plate centre O, axes x and y parallel and perpendicular to the trails, which are nearly straight.

Let P R Q be a trail ; and let

$$QR = RP = x, \quad OR = y.$$

Then distortion, which increases any ordinate y by $\Delta y \equiv Ar^2y$, elevates

$$P \text{ and } Q \text{ by } A(x^2 + y^2)y, \text{ and } R \text{ by } Ay^3 \text{ only,}$$

so that the depth of R below P Q is increased by

$$Ax^2y \text{ (approximately).}$$

If we take another trail, prq , higher up the plate, with ordinate y' , the effect of distortion on its curvature will be

Ax^2y' ; and thus the distortion affects the curvatures more and more as the trails travel up the plate in direct ratio to the distance.

Had we selected another law of distortion, say $\Delta r = Ar^5$, we should have had for the effect on curvature

$$\begin{aligned} & A(x^2 + y^2)^2y - Ay^5 \\ & = A(x^4 + 2x^2y^2)y, \end{aligned}$$

which varies more rapidly than the first power of y at a distance from the centre. If we take even powers of r for Δr , such as

$$\Delta r = Ar^2 \text{ or } Ar^4,$$

we cannot express the effect of curvature as a finite series of powers of x and y ; but the general run of the effect will be intermediate in character between that of two odd powers.

3. Thus we must be prepared to deal with trails in all parts of the plate—*i.e.* x and y may have values corresponding to the corners of the plate. We must now settle how big a plate we are going to measure. Suppose the plate something over 11° square, so that the coordinates of the corners, measured from the centre, are ± 0.1 in circular measure. The maximum value of successive powers and products of x and y is then shown in the following table:—

Circular Measure.	Arc	Circular Measure.	Arc.
$x = 0.1$	345'0	$x^4 = .0001$	20''7
$x^2 = 0.01$	34'5	$x^5 = .00001$	2'1
$x^3 = 0.001$	3'5	$x^6 = .000001$	2'1
		$x^7 = .0000001$	202

On such plates we may certainly reject x^7 and higher powers without any loss of accuracy—perhaps even x^6 ; but the computations will be carried out as far as x^6 .

Plates larger (in angular field) than this have been used; and we may readily estimate the effect of going beyond this limit. Suppose, for example, that the plate is four times this area—*i.e.* $23^\circ \times 23^\circ$. Then x^7 would represent in the corners $2''.6$. Now this is not a *very* large error to make as a maximum in a plate with this immense field, for unless the plate is of very large actual size, the scale must be comparatively small. For instance, the plate is unlikely to be larger than 23 inches $\times$ 23 inches, or 1 inch to the degree, one-third the scale of the Astrographic Catalogue plates; so that neglecting $2''.6$ in the corners is equivalent to neglecting $0''.9$ in the corners of Catalogue plates. Thus, though the formulæ developed in the following paragraphs are strictly accurate only for plates not larger than $11^\circ \times 11^\circ$, still they are probably accurate enough for *any* plate likely to be taken. If exceptionally

Thus

$$z = \left[\frac{1}{2}\xi^2 - \frac{\kappa}{8}\xi^4 + \frac{\kappa^2}{16}\xi^6 \right] \tan \delta.$$

Referring to the table in §3, we see that the maximum value of $\frac{1}{16}\xi^6$ is 0.01, and we may therefore reject this term. Since δ is small the term $\frac{\kappa}{8}\xi^4 \tan \delta$ is also small, and the formula

$$z = \frac{1}{2}\xi^2 \tan \delta$$

is very accurate even for large plates.

7. When D is not zero, but at the same time D and δ are less than 45° , then putting

$$\kappa = 1 - \sin^2 D \sec^2 \delta,$$

we have

$$\begin{aligned} \kappa u^2 + 2u - \xi^2 &= 0, \\ u &= \frac{1}{\kappa} \left\{ \sqrt{1 + \kappa \xi^2} - 1 \right\}, \\ &= \frac{1}{2}\xi^2 - \frac{\kappa}{8}\xi^4 + \frac{\kappa^2}{16}\xi^6, \\ z &= \left[\frac{1}{2}\xi^2 - \frac{\kappa}{8}\xi^4 + \frac{\kappa^2}{16}\xi^6 \right] \tan \delta. \end{aligned}$$

Since κ^2 is less than unity, the term $\frac{\kappa^2}{16}\xi^6$ can be rejected still more than before; and thus

$$z = \left(\frac{1}{2}\xi^2 - \frac{\kappa}{8}\xi^4 \right) \tan \delta$$

is an accurate formula for trails when δ and D lie between 0° and 45° .

8. But in experiments on optical distortion we do not need the absolute curvature of a trail, only the *difference* of curvatures between trails in the middle of the plate and elsewhere; and in many instances we do not require any formula at all. There are two distinct cases likely to occur in practice.

(A) When trails of the same star are taken in different portions of the plate; *i.e.* δ remains constant, and D varies from $\bar{\delta} + 5^\circ.7$ to $\bar{\delta} - 5^\circ.7$.

(B) When the plate centre is kept fixed and different stars allowed to trail over it; here D is kept fixed and δ varies from $D + 5^\circ.7$ to $D - 5^\circ.7$.

9. Take Case A first. Let $D = \bar{\delta} + \eta$. Then $\eta < 5^\circ.7$ and

$$\begin{aligned} \kappa &= 1 - \sin^2 (\bar{\delta} + \eta) \sec^2 \bar{\delta} \\ &= 1 - \tan^2 \bar{\delta} - 2\eta \tan \bar{\delta} \\ &= \kappa_0 - 2\eta \tan \bar{\delta}, \text{ say.} \end{aligned}$$

Let z_0 refer to a trail through the plate centre. Then

$$z_0 = \left(\frac{1}{2} \xi^2 - \frac{\kappa_0}{8} \xi^4 \right) \tan \delta$$

$$z = \left(\frac{1}{2} \xi^2 - \frac{\kappa_0 - 2\eta \tan \delta}{8} \xi^4 \right) \tan \delta ;$$

$$\therefore z - z_0 = +\frac{1}{4} \eta \cdot \tan^2 \delta \cdot \xi^4.$$

10. When $\delta = 45^\circ$ the maximum value of this expression, *i.e.* its value when $\xi = 0.1$, and $\eta = 0.1$ is $0''.5$ (see § 3); and thus for work in which we reject quantities less than $1''.0$ we may consider trails of the same star to have the same theoretical curvature all over the plate. For more accurate work we may use the following small table :—

Table of differences of curvature for trails of the same star in different parts of a plate.

If the trail be m degrees *south* of the plate centre, and $2n$ degrees long from end to end, and if the height of the ends above the middle be measured in seconds of arc and compared with a similar result for a trail of the same star through the plate centre, the excess of the former above the latter is $mn^4 \times 0.00001$ times the quantity tabulated below.

Decl. of Star.	Excess.	Decl. of Star.	Excess.
10°	$+0.3$	37.5°	$+4.8$
20	$+0.8$	40.0	$+6.0$
30	$+2.3$	42.5	$+7.1$
35	$+4.2$	45.0	$+8.4$

11. (B) If we expose the same plate to a series of stars of different declinations, without moving the plate centre, the calculation is not so simple. For the curvatures of two stars declinations δ_1 and δ_2 will differ by a quantity of a different order, *viz.* :

$$\frac{1}{2} \xi^2 (\tan \delta_1 - \tan \delta_2).$$

But using the table of Case A to reduce the curvature to the value it would have had if the trail had gone through the plate centre, we can now form tables for stars of different declinations giving the curvature of their trails when central on a plate. In fact we have to tabulate z_0 of the preceding case, *viz.* :

$$z_0 = \frac{1}{2} \tan \delta \cdot \xi^2 - \frac{1}{8} (\tan \delta - \tan^3 \delta) \xi^4.$$

The maximum value of the coefficient of ξ^4 , which vanishes when $\delta = 0^\circ$ and $\delta = 45^\circ$, occurs when $3 \tan^2 \delta = 1$, *i.e.* for $\delta = 30^\circ$; and its value is then 0.048 , the effect on z_0 being $1''.0$.

12. Thus if we are neglecting quantities less than $1''$, we may neglect this term altogether; *i.e.* combining this result with that of the last paragraph we may say that

The formula $z = \frac{1}{2} \tan \delta \cdot \xi^2$ gives the curvature of all star trails on plates not larger than $11^\circ \times 11^\circ$ wherever they may be on the plate, when δ lies between 0° and 45° with errors less than $1''$.

13. For more accurate work it is not difficult or troublesome to calculate the two coefficients of § 11; but a small table of $\frac{1}{8} (\tan \delta - \tan^3 \delta)$ will help, and then we can include both Cases A and B under one general formula, as follows:—

Let δ be the declination of a star and let its trail be $2n$ degrees long, and m degrees south of the plate centre. The height of the ends above the middle in seconds of arc is given by

$$z = 31'' \cdot 416 \cdot n^2 \cdot \tan \delta + (mn^4 \cdot A + n^4 B) \times 0.00001,$$

where A and B are given in the following table:—

$\delta =$	A	B	$\delta =$	A	B
0°	+0.0	0.0	25°	+1.4	43.8
5	+0.1	10.4	30	+2.3	46.2
10	+0.3	20.5	35	+4.2	42.8
15	+0.6	29.9	40	+6.0	29.9
20	+0.8	37.9	45	+8.4	0.0

14. *Refraction.*—We have now to examine what effect refraction has upon the curvature of a trail. In *Monthly Notices*, lvii. pp. 133, &c., it is shown that the effect of a refraction $\mu \cdot \tan Z.D.$ on a star whose coordinates are (x, y) when the coordinates of the zenith on the plate (supposed extended in the same plane so as to include the zenith) are (X, Y) is to increase x and y by Δx and Δy where

$$\Delta x = T(X - x) \quad \Delta y = T(Y - y)$$

and

$$T = \frac{\mu(1 + x^2 + y^2)}{1 + Xx + Yy}$$

(T being the same as $-t$ in the paper cited, p. 136).

Now $\mu = 57''$ approximately; hence for our plates the max. value of $\mu x^3 = 0'' \cdot 057$. There is no doubt we may reject μx^4 ; but μx^3 , especially if multiplied by 2 or 3, might introduce errors. Fortunately third powers of x and y do not occur in the present investigation, for we find the curvature of a trail by measuring the dip of the middle below the ends. If the effect of refraction be

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \&c.,$$

then at one end $x = +\xi$, and at the other $x = -\xi$, so that for the mean of the two the effect is

$$a_0 + a_2 \xi^2 + a_4 \xi^4 + \&c.,$$

and for the middle where $x=0$ the effect is a_0 . Thus the effect on the height of ends above middle is

$$a_2\xi^2 + a_4\xi^4,$$

and since we may certainly reject fourth powers we need only consider the term $a_2\xi^2$.

15. We assume that X and Y are not much greater than unity; i.e. that the plate is exposed at a Z.D. not much greater than 45° .

Then

$$\begin{aligned}\frac{1}{\mu}T &= (1+x^2+y^2)(1+Xx+Yy)^{-1} \\ &= [1 - Xx - Yy + x^2(1+X^2) + 2XYxy + y^2(1+Y^2)],\end{aligned}$$

and

$$\frac{1}{\mu}\Delta y = Y - y(1+Y^2) - xXY + x^2(1+X^2) + xy(X+2XY) + y^2(1+Y+Y^2).$$

In this we put from § 8

$$y = y_0 + \frac{1}{2} \tan \delta \, x^2,$$

and thus the coefficient of x^2 in $\frac{1}{\mu} \Delta y$ is

$$-\frac{1}{2} \tan \delta (1+Y^2) + (1+X^2) + y_0 \tan \delta (1+Y+Y^2).$$

The last term is equivalent to a term in x^3 , since y_0 is liable to the same maximum value as x . And if $Y=1$, $\tan \delta=1$ we may thus have from this term as much as $3 \times 0''\cdot 057 = 0''\cdot 17$. But under most conditions either $\tan \delta$ or Y would be small. We can, in fact, always expose a plate tolerably near the zenith if δ may be as much as 45° ; and hence we shall neglect the term $y_0 \tan \delta (1+Y+Y^2)$.

Thus the effect of refraction on curvature is expressed by

$$z = \mu[(1+X^2) - \frac{1}{2} \tan \delta (1+Y^2)]\xi^2.$$

16. Now if the plate centre be kept fixed as in Case B of § 11, and different stars allowed to trail across it, then X and Y remain the same and $\tan \delta$ varies from $D-5^\circ\cdot 7$ to $D+5^\circ\cdot 7$. Thus the difference in the values of z at the edges is

$$\begin{aligned}z_1 - z_2 &= \frac{\mu}{2} (\tan \delta_2 - \tan \delta_1) (1+Y^2)\xi^2 \\ &= \frac{\mu}{2} \sec^2 D \times 0\cdot 2 (1+Y^2)\xi^2 \text{ approx.}\end{aligned}$$

The factor $\sec^2 D (1+Y^2)$ is not much greater than unity; and since the maximum value of $\mu \times 0\cdot 1 \times \xi^2$ is $0''\cdot 056$, we may in general neglect this difference. Hence

If various trails be taken on a plate $11^{\circ}5' \times 11^{\circ}5'$ whose centre is kept fixed and exposed to a point within 45° of the zenith, the curvatures of all trails will be equally affected by refraction, if quantities under $0''.1$ be neglected.

17. If, however, the same star be used but the plate centre moved about so as to take different trails of it, then X and Y vary, but $\tan \delta$ remains constant. In this case

$$z_1 - z_2 = \mu \xi^2 [(X_1^2 - X_2^2) - \frac{1}{2} \tan \delta (Y_1^2 - Y_2^2)].$$

When X_1, X_2 , and Y_1, Y_2 are not very different this will be small. If the plate be exposed within 45° of the zenith, X_1, X_2, Y_1 and Y_2 are all less than unity; and since $\mu \xi^2$ has the value $0''.57$ at maximum, we are not liable to an error greater than this if we neglect the effect of refraction entirely. For some work this knowledge is sufficient. For instance, in his Memoir* on the determination of longitudes by photography, Captain Hills neglects all quantities less than $1''$; so that for his work the effect of refraction on the trails, whether of the same star or of different stars, may be neglected entirely, provided the plates be exposed at a Z.D. not greater than 45° .

18. For more accurate work we may examine the above expression a little more closely. The coordinates X and Y are determined from the equations

$$X = -\tan h \sin q \sec (P - q) \quad Y = \tan (P - q),$$

where

$$\tan q = \tan \lambda \cos h$$

and h is the hour angle of the plate centre, λ the colatitude.

Now we shall suppose the plate exposed within a couple of hours of the meridian either way, which gives four working hours. Then the maximum value of h is $.5$; of h^2 $.25$; and of h^4 $.06$. Thus as an approximation we may take

$$q = \lambda - \frac{1}{4} h^2 \sin 2\lambda$$

$$X = -h \sin \lambda \sec (P - \lambda) - h^3 \left\{ \frac{1}{3} - \frac{1}{2} \cos \lambda \cos P \sec (P - \lambda) \right\}$$

$$Y = \tan (P - \lambda) + \frac{1}{4} h^2 \sin 2\lambda \sec^2 (P - \lambda) + \text{fourth powers.}$$

Thus, if we neglect h^4 , we may write

$$X^2 = h^2 \sin^2 \lambda \sec^2 (P - \lambda)$$

$$Y^2 = \tan^2 (P - \lambda) + \frac{1}{2} h^2 \sin 2\lambda \tan (P - \lambda) \sec^2 (P - \lambda).$$

Now $P - \lambda$ is the meridian Z.D. of the plate centre. But it is readily seen that we shall not make a sensible error if we substi-

* *Memoirs of Royal Astronomical Society*, vol. liii. p. 117.

tute for this $p - \lambda$, the meridian Z.D. of the star. Call this ζ . Then

$$\begin{aligned} Y_1^2 &= \tan^2 \zeta + \frac{1}{2} h_1^2 \sin 2\lambda \tan \zeta \sec^2 \zeta \\ Y_2^2 &= \tan^2 \zeta + \frac{1}{2} h_2^2 \sin 2\lambda \tan \zeta \sec^2 \zeta \\ \therefore \frac{1}{2} \tan \delta (Y_1^2 - Y_2^2) &= \frac{1}{4} (h_1^2 - h_2^2) \sin 2\lambda \tan \delta \tan \zeta \sec^2 \zeta. \end{aligned}$$

Now for ordinary latitudes, such as those of Europe, the product $\tan \delta \tan \zeta$ is small. For stars near the equator δ is small, and for stars of declination 45° , ζ is small. The maximum value of $\tan \delta \tan \zeta$ subject to the condition

$$\delta + \zeta = 90^\circ - \lambda = \phi \text{ (the latitude)}$$

is $\tan^2 \frac{1}{2} \phi$; the value of which is as follows :—

Latitude =	30°	40°	50°	60°
$\tan^2 \frac{1}{2} \phi$	·07	·13	·22	·34

Also $h_1^2 - h_2^2$ is not greater than ·25. Hence we can neglect the terms depending on Y .

As regards $X_1^2 - X_2^2 = (h_1^2 - h_2^2) \sin^2 \lambda \sec^2 \zeta$, the maximum value of this term, when $h_2 = 0$ and $h_1^2 = \cdot 25$, is about ·25, and the effect on the refraction is about 0''·14. We may fairly neglect such small quantities except for the most refined investigations.

Thus in this case also the trails are equally affected by refraction in all parts of the plate, if the exposures be confined to the two hours preceding and the two hours following the meridian and the star be between the equator and the zenith.

Conclusions.

(a) Let a trail of a star declination δ be taken on a plate of field $11^\circ \cdot 5 \times 11^\circ \cdot 5$, whose centre is in declination D ; and let (X, Y) be the standard coordinates of the zenith on the plate. Then the equation to the trail in standard coordinates (ξ, η) expressed in circular measure, is

$$\begin{aligned} \eta &= \tan (\delta - D) \\ &+ \frac{1}{2} \tan \delta \cdot \xi^2 \\ &- \frac{1}{8} (\tan \delta - \tan^3 \delta) \xi^4 \quad (\text{max. value } 1'' \cdot 0) \\ &- \frac{1}{4} \tan^2 \delta \cdot \tan (\delta - D) \cdot \xi^4 \quad (\text{max. value } 0'' \cdot 5) \\ &+ \mu [(1 + X^2) - \frac{1}{2} \tan \delta (1 + Y^2)] \xi^2 \quad (\text{max. value about } 0'' \cdot 6). \end{aligned}$$

(b) If the exposures be at Z.D. less than 45° , and within two hours of the meridian, and if we take the *difference* between trails and not a single trail only, then the fifth line, due to refraction, may always be neglected, even for measurements of considerable precision.

(c) Under the conditions specified in (b) the third and fourth lines are quickly calculated by the table in § 13.

(d) If, further, the trails be of the same star, we need *only the fourth line*, as given by the little table of § 10.

(e) If we neglect quantities less than $1''\cdot 0$, then the curvatures of all trails on a plate exposed near the meridian between $D=0^\circ$ and $D=45^\circ$ should be the same, unless there is optical distortion.

Remarks on the Paper by Professor W. Schur, together with determination of the Diameter and Polar Compression of the Planet Mars from Observations with the Repsold Heliumeter of the Remers Observatory, Bamberg, and with the Breslau Heliumeter at the Observatory, Strassburg, in 1879. By Dr. Ernst Hartwig.

(Communicated by the Secretaries.)

In the March number of the *Monthly Notices* (p. 330) Professor Schur has communicated a series of heliometer measurements of the polar and equatorial diameters of the planet *Mars*, from which he deduces a polar compression of a fiftieth. In that discussion no reference is made to the probable errors of the results which are said to be of a greater weight than the earlier researches because an ocular reversing prism was used. Computing the mean errors for the single measures and for the results of one day I have found them (in spite of "images being steady") greater than they were in the measurements made by the same observer with the little Breslau heliometer at Strassburg in 1877. The mean error of a diameter reduced to mean distance of the planet *Mars* from the Sun is for the polar diameter $\pm 0''\cdot 112$, for the equatorial $\pm 0''\cdot 094$ (in mean distance Sun—Earth $\pm 0''\cdot 170$ and $\pm 0''\cdot 143$), therefore for the measured diameter $\pm 0''\cdot 23$ and $\pm 0''\cdot 19$, or for a distance of *Mars* the same as in 1877 $\pm 0''\cdot 454$ and $\pm 0''\cdot 378$, the corresponding mean errors in 1877 having been for the measures made by Dr. Schur with the Breslau heliometer $\pm 0''\cdot 208$ and $\pm 0''\cdot 207$.

The three days of 1899 give a mean difference of $0''\cdot 170$ between both directions (polar and equatorial) for the diameter in the mean distance between Sun and Earth, and the mean error of it is $\pm 0''\cdot 128$, because the single difference has the mean error $\pm 0''\cdot 222$. The measures in 1896 are better, the mean error for the result of a day being (polar diameter) $\pm 0''\cdot 031$ and (equatorial diameter) $\pm 0''\cdot 064$ in mean distance of the Earth from the Sun, whence we find for the single difference the mean error $\pm 0''\cdot 071$, and for the mean, $0''\cdot 205$, of the four differences the mean error $\pm 0''\cdot 036$. But the measures in 1899 in view of the great uncertainty do not prove that the compression is in conflict with

that which Hermann Struve has calculated from his researches on the motions of the apsides of the satellites *Phobos* and *Deimos*; and the measures in 1896 disagree with those in 1899 made by myself, which are independent of errors in estimation. I have also measured the diameter of *Mars* with the great Repsold heliometer of the Remeis Observatory since 1890, using the ocular reversing prism to eliminate the personal errors in measuring diameters of discs in different directions with respect to the vertical line. The bad conditions of atmosphere have prevented my getting more than one opportunity for measuring in both directions in each opposition 1890 and 1899. The measures, all made with apparent vertical motion of the images by means of the ocular reversing prism and corrected for defect of illumination, which I have directly computed in Bessel's manner, are the following:—

Date, 1890 (3 May 27).	Mean Time, Bamberg.	Position-Angle.		Measured Diameter.	Diameter at Mean Distance Sun—Earth.
		Observed.	Computed.		
May 6	^h 12 ^m 46	54° 2'	36° 3'	16" 522	9" 222
	13 0	144° 2'	126° 3'	16° 904	9° 435
	13 14	144° 2'	126° 3'	16° 828	9° 393
	13 29	54° 2'	36° 3'	16° 773	9° 362
¹⁸⁹⁴ (3 Oct. 20).					
July 24	14 57	156° 8'	126° 8'	12° 757	9° 338
Aug. 26	16 52	156° 9'	129° 9'	16° 351	9° 196*
Sept. 14	13 55	156° 3'	130° 2'	19° 437	9° 424
¹⁸⁹⁹ (3 Jan. 18).					
Feb. 4	9 48	350° 9'	345° 3'	13° 356	9° 278
	10 2	260° 9'	255° 3'	13° 447	9° 340
	10 15	260° 9'	255° 3'	13° 533	9° 401
	10 37	350° 9'	345° 3'	13° 333	9° 262
Feb. 21	10 42	345° 2'	343° 3'	11° 629	9° 205†

whence we have

	Polar Diameter.	Equatorial.	$2a - 2b$	$\frac{a-b}{a}$
1890	9" 292	9° 414	0° 122	$\frac{1}{77}$
1899	9° 270	9° 370	0° 100	$\frac{1}{94}$

Neglecting the deviation in areographic latitude $\pm$ the mean of the differences is 0''·111. The mean error of a measure for the polar diameter from observations 1890, 1894, and 1899 is $\pm 0''\cdot 068$, and if we assume the same mean error for the equatorial diameter, we get for the mean error of a single difference $\pm 0''\cdot 096$ and

* In daylight.

† Only measured for position-angle.

‡ The areographic latitude 90° , in the paper of Professor Schur, is not right, because the middle of the disc of *Mars* was $13^\circ 6'$ north of its equator at opposition, January 18.

of the mean of the two differences $\pm 0''.068$, essentially less than the value of the difference itself; therefore a polar compression seems to exist and not to be in too great discordance with the theoretical value. In the *Ast. Nach.* 2272 I drew attention to the extremely good opportunity afforded by the opposition of 1879 for measuring the polar and equatorial diameter of *Mars*, each in both vertical and horizontal directions at eastern and western hour-angles. I have made a large series of measurements with the Breslau heliometer at Strassburg, the results of which will appear shortly in the *Astron. Nachrichten*. Herewith I have the honour to communicate the results of measurements of both diameters, obtained near the opposition of 1879. By v and h are denoted vertical and horizontal direction of the rotation axis, and by $\times$ an inclination of nearly 45° , when *Mars* was passing the meridian.

Date, 1879.	Mean Time, Greenwich.	Rotation Axis.	Diameter—		Diameter at Mean Distance, Sun—Earth.	
			Polar.	Equatorial.	Polar.	Equatorial.
Oct. 5	h					
	10.8	v	17.137	17.280	9.394	9.473
	16.0	h	17.015	17.501	313	579
	7	v	17.351	17.412	376	411
	13	v	18.047	18.098	378	405
24	10.6	v	18.940	19.170	322	435
Nov. 7	11.8	$\times$	19.319	19.513	338	432
	8	h	19.487	19.673	434	524
	9	v	19.290	19.553	355	482
	14	$\times$	19.124	19.319	411	507
	27	v	17.830	17.791	476	457
		$\times$	17.598	17.773	365	457
	28	v	17.400	17.690	323	479
		$\times$	17.208	17.641	230	462
		h	17.266	17.487	270	388
	29	$\times$	17.102	17.396	253	412
		h	17.360	17.126	401	275
Dec. 2	13.7	h	16.865	16.872	374	376
	7	v	16.072	16.299	342	472
Mean					9.353	9.446
Mean error					± 0.015	± 0.016

The measures reduplicated on the same day are made in different positions of the disc relative to the eye of the observer, and may be considered as independent. No systematic discordance occurs between the measurements in the two (or three) directions of the two diameters with the vertical line. For we have

Polar Diameter.			Equatorial Diameter.		
v	h	x	v	h	x
9''394	9''313	9''338	9''579	9''473	9''432
376	434	411	524	411	507
378	270	365	388	405	457
322	401	230	275	435	462
355	374	253	376	482	412
476				457	
323				479	
342				472	
9''371	9''358	9''319	9''428	9''452	9''454
$v-h = +0''.013$			$v-h = -0''.024$		

in both cases the difference between vertical and horizontal direction being smaller than the probable error of it.

The differences "equatorial minus polar" for the three directions are

$2a-b$	$\frac{a-b}{a}$
$v + 0''.057$	1 : 166
$h + 0''.094$	1 : 101
$x + 0''.135$	1 : 70

Hence the figure of the planet, when the axis of rotation is inclined at 45° to the vertical line, seems to the observer to be different from that in the other positions. But the result depends chiefly upon the two measures of the polar diameter made on November 28 and 29, which are the smallest of the whole series. I believe the mean of all the measures to be free from personal errors arising from difference in the position of the rotation axis in regard to the vertical line. Hence we have the difference between the polar and equatorial diameters in mean distance of the Earth from the Sun $= 0''.093$ with the mean error $\pm 0''.021$, and the polar compression $= \frac{1}{10.2}$, agreeing well with the result obtained with the great Repsold heliometer of the Remeis Observatory, $0''.111$, i.e. $\frac{1}{8.5}$. The mean of the measurements of the polar diameter with the latter heliometer on four days, viz. $9''.331$ (mean error $\pm 0''.034$), is also in agreement with the result above, $9''.353$ (mean error $\pm 0''.015$). The higher power of the Bamberg heliometer just compensates in these measures for the greater apparent diameter of the disc as measured with the Breslau heliometer in the opposition of 1879. The value $0''.1$ for the difference between the polar and equatorial diameter of *Mars* in mean distance of Earth from Sun is doubtless not far from the truth.

Observations of Mars made at Mr. Crossley's Observatory, Bermer-side, Halifax, during the Opposition 1898-9. By J. Gledhill.

I.

Notes on the Markings Seen on the Disc.

The following observations of *Mars* were made with the 9-inch Cooke Equatorial Refractor (the new triple object glass). The powers used were 240 and 330; the former as that most generally useful, the latter on the very few exceptionally good nights. The planet was carefully examined—indeed, watched almost continuously, often for several hours—on every clear night from 1898 December 19 to the end of March 1899, in all some forty nights. The definition was never continuously good, and the seeing and identification of the features often called for much patient gazing. The limb and the terminator were on every occasion most carefully examined for irregularities of form. No such were ever surely seen except on one night, the one occasion when the perfect stillness of the planet allowed of the use of powers 330 and 470. The southern edge of the N. polar ice-cap was often examined, but no trace of any breaks or projections was ever seen. To a very large extent, no doubt, these failures were simply a measure of the bad observing conditions experienced here during the winter. The projection seen for more than an hour on 1899 January 24 would certainly have escaped detection on an average night owing to the undulations on the limb.

1898 December 19, 11^h to 12^h , λ , the longitude of the central meridian, $=295^\circ$ to 310° . Bad definition, p limb very bright, the bright lune extending inwards up to the Kaiser Sea; f limb dull; the N. polar ice-cap large and white. The dusky Delambre Sea extends from the S. edge of the ice-cap nearly to the N. point of the Kaiser Sea. All the region to the east of the Kaiser Sea (between it and the f limb) was of a warm tinge.

1898 December 20, 11^h to 12^h , $\lambda=286^\circ$ to 301° . There was a bright region about the S. pole of the disc—probably Lockyer Land. It was bright and of a pale yellow tint. As always, the N. portion of the Kaiser Sea was the darkest portion of that feature, and also perhaps one of the darkest parts of the disc. There was a little warm colour in the region p , the Kaiser Sea (Herschel I. Continent), and a deeper tint in the region following it (Beer Continent). The bright lune at p limb was much narrower than on the 19th. Delambre Sea was seen, but not well.

1898 December 22, 11^h , $\lambda=268^\circ$. Lockyer Land, the Kaiser Sea, the grey space between the two, Delambre Sea and the coloured continents of Herschel I. and Beer, were seen as on the 20th.

The p and f limbs differed very little in brightness: the colour did not run quite up to the p limb as it did to the f limb. The bright lune on the p limb was narrow. At 12^h , $\lambda=283^\circ$, the Kaiser Sea was about central. The dark fringe on the S. edge of the N. ice-cap was seen, but not clearly. The p limb was perhaps not quite so bright as the f limb. At 13^h , $\lambda=297^\circ$, the Delambre Sea was better seen. Lockyer Land was bright. At 14^h Nasmyth Inlet and Laplace Land were seen, as well as Herschel Strait and Phillips Island.

1898 December 24, 11^h , $\lambda=250^\circ$. The Kaiser Sea was not far from the f limb; the deepest colour lay to the west of it (Herschel I. Continent), a paler tint between it and the f limb. The p limb was thought to be less bright than the f limb. A faint marking was seen on Herschel I. Continent a little to the E. and S. of Fontana Land. Delambre Sea was seen.

1898 December 27, 12^h , $\lambda=238^\circ$. Again the f limb was thought brighter than the p . All other features as on the 24th.

1899 January 4, 11^h , $\lambda=153^\circ$. The only feature seen in the N. portion of the disc was Oudemans Sea. The bright region round the S. pole of the disc was probably Webb Land. Of the E. and W. limbs the p was the brighter. Of course Maraldi Sea was seen, but the definition was not good enough to show Trouvelot Bay and Noble Cape.

1899 January 11, 8^h , $\lambda=48^\circ$. With the exception of the grey band about the S. pole of the disc (the De Totingne Sea &c.) the only marking seen was Airy Sea with a portion of Campani Sea to the N. of it. The bright lune of the p limb was wide.

1899 January 13, 7^h , $\lambda=16^\circ$. Knobel Sea, the bright Mädler Continent, the grey De la Rue Ocean and the bright Jacob Land near the S. edge of the disc were well seen. The bright lune of the p limb was very bright and extended far inwards, say one fifth or one-sixth of the radius of Mars. There was also a considerable brightening of the warm-toned area to the east of the central meridian and near the f limb. That portion of Campani Sea just N. of Knobel Sea was darker than the latter, and it was in contact with the ice-cap. Burton Bay was seen.

1899 January 19, 7^h , $\lambda=323^\circ$. The Kaiser Sea was not far from the p limb; there was no warm colour between it and that limb. There was a faint warm tinge along the f limb, and the bright lune on the p limb was a well-marked feature. Dawes Forked Bay was seen, but not Burton Bay. Delambre Sea lay along the S. edge of the ice-cap and was darkest opposite the Kaiser Sea. At 8^h the Kaiser Sea was much fainter, being near the p limb. The darkest feature of the disc was Herschel II. Strait and the two forks. Phillips Land, Arago Strait, Knobel Sea, Kunowski Land, and Jacob Land were seen at 9^h . The Kaiser Sea remained faintly visible when it seemed (*i.e.* the eastern boundary of it) but a line close to the p limb. Nasmyth

Inlet and Lassell Sea were never seen except when the observing conditions were good.

1899 January 24, 7^h, $\lambda = 280^\circ$. The Kaiser Sea was near the central meridian. Delambre Sea was seen. A small bright roughly circular spot was seen close to the western side of the Kaiser Sea near where the equator of Mars cut it. Soft warm colour overspread all the region E. and W. of the Kaiser Sea. The width of the bright lune on the *p* limb was about one-fifth of the radius of the planet. At 9^h Nasmyth Inlet and Lassell Sea were seen : of these two features the portions best seen on all occasions are the hump of the former near the N. point of the Kaiser Sea, and the whole of the southern projection called Lassell Sea. Dawes Forked Bay and Phillips Land were seen.

1899 January 25, 7^h, $\lambda = 271^\circ$. The bright lune at the *p* limb, the much less bright terminator, the Kaiser Sea and the ice-cap, all as on the 24th. The Kaiser Sea was central about 8^h. At 8½^h Lassell Sea was seen occasionally. Dawes Forked Bay was seen, faint and near *f* limb : the two forks or inlets seen as one—*i.e.* definition was not good enough to separate them. Up to 12^h the definition was very poor. Projections &c. on limb and terminator were carefully looked for.

1899 January 31, 10^h. The Kaiser Sea was near the central meridian ; the warm colour was deeper to the W. than to the E. of it. Very cloudy night.

1899 February 1, 7^h, $\lambda = 209^\circ$. Oudemans Sea was about central ; close to its eastern or *f* side was the bright circular region called Fontana Land. The eastern edge of Oudemans Sea was darker than the western. At the first glance this sea might be mistaken for the Kaiser Sea : its shape was triangular with the point downwards, *i.e.* to the N. This northern-pointed portion did not appear to be connected with Schröter Sea to the N. of it. The *p* limb was very bright ; the terminator very much less bright.

At 8½^h Delambre Sea was seen near the *f* limb and resting on the ice-cap. At 9^h the Kaiser Sea lay close to the *f* limb, faint, but easily seen ; it was of about the same grey tone as the seas about the S. pole of the disc. At 10^h Webb Land and Burckhardt Land were seen as one continuous broad light-coloured region, *i.e.* the narrow portion named Niesten Isthmus was not noticed.

1899 February 14, 5^h. Airy Sea was not far from the *p* limb : it appeared as a dusky region resting on the S. edge of the ice-cap (N.), and a dusky wisp or two were seen in the central portion of the disc. The S. polar region was bright. At 7^h a portion of De la Rue Ocean was near the *p* limb and was the darkest marking of the disc. A very slight dusky fringe lay along the S. edge of the ice-cap : the *p* limb very bright : the terminator dull. This S. edge has been very carefully examined on every fine night, but no projections or irregularities in the curve were seen.

1899 February 16, 7^h , $\lambda=76^\circ$. Airy Sea faint; Campani Sea, to the N. of it, was the darkest feature of the disc. A faint dusky sweep was seen to the S.E. of Airy Sea: it was probably the faint grey region separating the Mädlar and Secchi Continents in Green's map. Christie Bay was of course well seen.

1899 February 17, 6^h , $\lambda=52^\circ$. Campani Sea was the darkest marking of the disc; it was very dark. Airy Sea was not so dark, its southern portion being still less dark and extending southwards to the equator of the disc. Christie Bay was dark, and the dusky sweep to the S.E. of Airy Sea was again seen. At 7^h and 8^h Jacob Land was bright and coloured like Secchi Continent.

1899 February 21, 7^h , $\lambda=31^\circ$. The southern boundary of the Polar cap was a dark fine line: on it lay a dark form, Knobel Sea, very dark in its northern portion and growing fainter to the South. A faint sweep of grey was seen to the N.E. of Knobel Sea at the eastern end of Mädlar Continent. The preceding end of the broad band (De la Rue Ocean &c.) seemed to stop short suddenly a good way from the p limb, while at the terminator this same broad grey band extended quite up to the edge of the visible disc. De la Rue Ocean was not nearly so dark as the Northern part of Knobel Sea and the Campani Sea to the north of that.

1899 February 22, 7^h to 9^h . Knobel Sea, the bright division between it and Campani Sea, Campani Sea, Christie Bay, all seen as on the 21st. The bright Rosse Land was not seen. Hall Land probably seen: *i.e.* a small bright region was seen not far from the S. pole of the disc, which was probably Hall Land.

1899 February 23. Looked carefully for the bright region (see Green's map) called Rosse Land, but did not see it. The three bays, Dawes Forked Bay and Burton Bay, were well seen.

1899 February 24, 7^h to 9^h . The regions seen were the three bays (Dawes and Burton), Knobel Sea, Leverrier Land, Lassell Sea, Phillips Island, Arago Strait, Jacob Land, Hall Island, and the Kaiser Sea.

1899 February 25, 7^h to 8^h . The Kaiser Sea lay near the p limb; it became gradually fainter as it neared the limb, and at last broke the bright line of the limb, *i.e.* the very bright line of the limb was less bright at that part than to the north and south of it. Knobel Sea, Leverrier Land, Lassell Sea, Nasmyth Inlet, the three bays, Phillips Land, and Laplace Land were seen. A bright spot was noticed near where Proctor Cape is marked on Green's map. The fringe along the south edge of the polar cap was faint and narrow.

1899 February 26, 7^h to 9^h . The Kaiser Sea was again watched as it approached the bright lune of the p limb, and again that portion of the lune was dimmed. The bright Rosse

Land was looked for but was not seen. All the features seen on the 25th were again easily seen. Lassell Sea could be seen continuously. Nasmyth Inlet was seen now and then for a moment : it was darker but smaller than Lassell Sea. Or, it may be said, that the fainter broad inlet between the N. end of Lassell Sea and the N. end of the Kaiser Sea was seen with some difficulty ; while the dark protuberance at the *p* end of the inlet was much more easily seen.

1899 February 27, 7^h. The Kaiser Sea, the three bays, Nasmyth Inlet, Lassell Sea, and Knobel Sea, were seen. It was not a good night.

1899 March 1, 7^h to 12^h. The phenomena attending the approach of the Kaiser Sea to the *p* limb were seen as on February 26. The long narrow Nasmyth Inlet and Lassell Sea were steadily seen, also Lockyer Land, Phillips Land, Knobel Sea, and the three bays. Lassell Sea and the protuberance at the *p* end of Nasmyth Inlet were much darker than the intermediate portion (*i.e.* the portion over which stands the name Nasmyth Inlet in Green's map). The same remarks as to objects seen &c. apply to March 2, 7^h to 10^h.

1899 March 2, 9^h. The Kaiser Sea was near the *p* limb, Knobel Sea near the terminator. The Forked Bays, Delambre Sea, and the bright Phillips Land were seen and identified. When the Kaiser Sea reached the bright lune on the *p* limb, the latter was shorn of much of its brightness. At 7^h Lassell Sea and Nasmyth Inlet were seen.

1899 March 15, 7^h to 8^h. Good definition : power 330 : some mist. The features seen may be described thus :—A small bright region round the S. pole of the disc, probably Webb Land ; the broad, greenish grey band extending from limb to terminator, *i.e.* Maraldi Sea, &c. &c., and which extended quite up to the terminator but became very faint near the *p* limb ; the broad (one-fifth to one-fourth of the radius) bright lune along the *p* limb : a dusky form or feature not unlike the Kaiser Sea, *i.e.* a triangular form, point to the north base to the south, detached from the broad band to the south (Maraldi Sea, &c.), identified as Oudemans Sea : to the north again, resting on the south edge of the N. polar ice-cap lay Schröter Sea : Secchi Continent lay to the west, a warm yellow region, and Herschel I. Continent, of a similar tint, lay to the east of the central Oudemans Sea ; the small, bright, roundish Fontana Land was seen well. The definite outline, darkness, and general similarity of Oudemans Sea to the Kaiser Sea, and its extreme unlikeness to the region so named on Green's map, made this a very interesting object. On the 16th the same features were seen in the same way as on the 15th.

1899 March 16, 10^h to 11^h. The triangular form like the Kaiser Sea, again well seen. The central portion is the darkest. Its E. and W. edges were well defined. The base or S. side of the triangle faded away into invisibility.

1899 April 16, 8^h, power 330. The features seen were a small, bright, yellow spot at the S. pole of the disc, the broad grey band of seas &c. extending from limb to terminator, the Kaiser Sea near the terminator, the N. polar cap and the dusky Delambre Sea on its southern edge.

1899 April 17 ; a few minutes after sunset : definition good ; powers 330 and 470. That part of the *p* limb which an hour later will be a bright lune is now of a bright warm colour, while the rest of the disc (the great southern band excepted) is of a dull yellow tint. A long grey form lay near the *p* limb ; no doubt it was Oudemans Sea. Delambre Sea appeared, as usual, as a grey form resting on the south edge of the N. polar cap, being darkest close to the cap. Fontana Land was not seen.

1899 April 18, 7^h. Oudemans Sea was about midway between the central meridian and the *p* limb. Delambre Sea &c. as on the 17th. At 8^h on the 19th the same features were seen as on the 18th, only a little more westward, *i.e.* nearer the *p* limb.

1899 April 19, 8^h. Oudemans Sea near the *p* limb. Delambre Sea seen.

2.

Note on a Projection Seen on the Terminator.

1899 January 24. Calm, misty, definition good, image of planet quite steady. On every clear night during this opposition of Mars the limb and the terminator were carefully scrutinised for projections, &c. About 9¹/₂^h a small, seemingly round projection was seen on the terminator : it was about as bright as the disc near the terminator and reminded one of a Jovian satellite when in exterior contact. As to diameter, it was perhaps about 0''·5, and reckoning the position angle from the middle of the outer edge of the N. polar cap it was about 150°, *i.e.* it was far up towards the S. pole of the disc in the *sf* quadrant. It remained visible for about an hour, and was invisible at 11^h and 12^h. This was the one night of really fine definition out of the forty on which the planet was observed.

In the above notes *east* and *west* are used in the ordinary astronomical (not areographical) sense.

Note on the Constitution of Saturn's "Crape" Ring.

By E. M. Antoniadi.

Some three years ago the idea was suggested by the writer that the phenomena presented by *Saturn's* inner "dark" ring might be explained by assuming the albedo of its particles to be equal, or at any rate comparable to that of the bright rings, in which case the shading marking the projection of this ring on the planet would be merely the shadow cast by that swarm of meteors.*

Before examining the results arrived at by an application of the deductive method to this interpretation, it would be useful to start with a sound notion of the distribution of matter in the "crape" ring. Professor Barnard's observations of the occultation of *Iapetus* in 1889,† and of the physical appearance of the planet in 1894,‡ have shown this ring to be "very thin at its inner edge," and growing "much denser where it joins the bright ring." True, a reversal of these appearances has been often noted with smaller instruments, the "dark" ring showing itself, at the ansæ, brighter towards the globe of *Saturn*. But this effect is evidently of a purely subjective character, the juxtaposition of the bright ring dwarfing to invisibility that part of the "crape" ring lying in its immediate vicinity. Besides, the fact that the "dark" ring is more transparent to the planet's limb towards its inner edge than close to the bright ring is a striking confirmation of its greater rarefaction near *Saturn*.

This point once established, the hypothesis above enunciated leads us to the following conclusions :—

1. Inasmuch as the heliocentric latitude of *Saturn* can attain the value of $2^{\circ} 30'$, the outline of the dusky shadow projected on the globe would not usually be a rigorous continuation of the "nebular" ansæ.

(a) Should the Sun be higher above the plane of the ring-

* *Journal of the British Astronomical Association*, vol. vii. pp. 241, 242.

† *Monthly Notices*, vol. I. January 1890.

‡ *Ibid.* vol. I. May 1895.

system than the Earth, the breadth of the shadow across the planet would shrink along the minor axis.

(b) Should the Sun be lower above the same plane than the Earth, the breadth of the shadow would be increased by the additional shadow of the inner edge of the bright ring. But the darkness of this latter shadow, viewed, as it would be, through the thickest (outer) part of the light-scattering swarm, would be considerably attenuated.

2. The real intensity of the "crape" ring's shadow being an inverse function of the Sun's altitude above the plane of the rings, the transparency of the "dark" ring ought to diminish with the closing of the system. For the perspective grouping of the particles would, in this case—

(a) Mask more effectively the planet's limb ;

(b) Whose intensity would be further reduced by the strengthening of the particles' shadow, consequent on their closer apparent grouping.

Now observation confirms both these deductions. With reference to the latter, we find Proctor saying :—"As the ring-system closes up, the distinction between the dark ring and the neighbouring bright ring becomes less marked, the dark ring appears greyish or slate-coloured, the traces of division less distinct (or less frequently to be noticed), while the outline of the planet is either not seen at all through the dark ring, or only seen with difficulty and indistinctly." *

That the first conclusion is also in accordance with experience the writer only recently found out during a visit paid to the Bibliothèque Nationale, Paris. While consulting there the literature on the subject, he came across a statement of Dawes' in the *Monthly Notices* for January 1851, p. 52, running thus :—"I have always† observed that the upper (southern) and more distant portion of the obscure ring is more plainly seen than the corresponding portion on the side nearest to the Earth, and also that the projection of it at its minor axis is considerably narrower than accords with its breadth at the major axis."

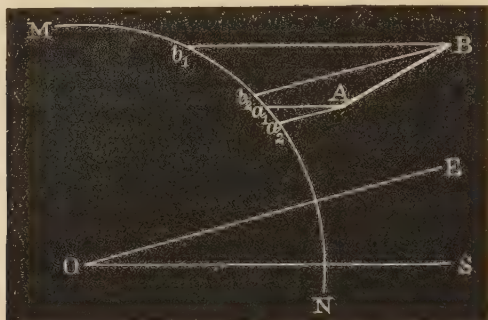
The first part of this sentence is a confirmation of Dr. Barnard's results, above alluded to, while the latter half might be accounted for, by the theory we are examining, in the following manner :—At the time of the observations of Dawes—let us say, 1851 January 1—*Saturn* had the following elements :—

Heliocentric latitude (almost at its maximum negative value)	...	...	...	— 2° 29'
Heliocentric longitude	...	...	...	20° 24'
Visible surface of ring system	...	...	...	Southern

* *Old and New Astronomy*, p. 632.

† Dawes having seen the "crape" ring for the first time on 1850 November 2, the word *always* cannot embrace more than a few weeks in the past.

Let MN in the annexed figure be the outline of *Saturn*, O its centre, AB the "crape" ring, supposed to consist of particles at



least as bright as the planet's surface, OE the direction of the Earth, OS that of the Sun at the time of Dawes' observation, the angle EOS being, for the sake of clearness, grossly exaggerated. Then the point A of the "crape" ring casts its shadow on the planet at a_1 (Aa_1 being parallel to SO), B at b_1 ; the arc a_1b_1 marking the breadth of the projection of the shadow on the globe. But to the observer on the Earth, the point A is projected at a_2 (Aa_2 being parallel to EO), B at b_2 ; and it will be seen that the shadow ought to appear shrunk along the minor axis, inasmuch as Dawes was seeing shaded, through the gaps separating the particles along AB, the segment $a_1b_2 < a_2b_1$, the arc a_1a_2 suffering no obscuration through the projection on it of particles whose albedo is fully comparable to its own.

Were the "crape" ring to be really a *dark* ring, we ought whenever the heliocentric latitude of *Saturn* is considerable, to be enabled to distinguish the dusky projection from the shadow it would cast on the globe. Such a difference of shade in the band crossing the planet was sought for by the eagle-eyed Dawes in 1852, but in vain. And it is evident that this very failure, which is unaccountable by any idea associated with a dark ring, is a forced corollary of the theory above examined.

In 1884, M. Trouvelot attacked this subject in his valuable paper entitled *Sur la Variabilité des Anneaux de Saturne*, and published in the *Bulletin Astronomique*.* He also thought that the phenomena observed by Dawes and himself could be explained by the "crape" ring's shadow. But the possibility that the albedo of the individual particles of this ring might be identical with that of the bright rings, which is the corner-stone of the writer's interpretation, does not seem to have dawned in M. Trouvelot's mind, for not only does he not make the slightest allusion to this assumption, but furthermore seems unmistakably to espouse

* Numbers for November 1884 and January 1885.

the view that the "crape" ring is really a dark ring, as the following quotation from his paper proves beyond doubt:—"I then attributed," he says, "the phenomenon" (narrowing of the "dark" ring at lesser axis) "to an effect of irradiation of the light of *Saturn's* globe overstepping the material particles composing the border of this ring. Although it might be probable that irradiation must cause a reduction in the diameter of the particles of the ring projected on the globe, I now think, however, that the phenomenon just described results from another cause." As irradiation is capable of affecting the diameter of dark bodies only when put in juxtaposition with a bright one, M. Trouvelot obviously considered the particles of the "crape" ring to be darker than the globe. But then, without irradiation, the segment a_1a_2 in the preceding figure ought to be dark, not bright as the planet, as Dawes actually saw it. And thus M. Trouvelot's interpretation is shown to be in opposition to observation.

Irradiation doubtless affects the breadth of the shadow cast on the globe by the "crape" ring, but to a slight extent only, as the intensity of such shadow is also slight. Inasmuch, however, as the luminosity of the planet is greatest about its centre, waning very rapidly towards the limb, the effect due to irradiation would not be uniform, attaining its maximum at the minor axis, its minimum in the vicinity of the limbs, a circumstance which would tend to exaggerate the apparent concavity of the shadow's outline with regard to the centre of *Saturn*.

Observations of the Satellite of Neptune from Photographs taken at the Royal Observatory, Greenwich.

(Communicated by the Astronomer Royal).

A number of photographs of Neptune and his satellite (twenty-two in all) have been obtained since December 23 with the Thompson Equatorial, using either the 26-inch refractor or the 30-inch reflector. From January 26 an occulting shutter immediately in front of the plate has been used to screen the planet during the greater part of the long exposure on the satellite, a series of very short exposures (usually twenty of one second each) being given for Neptune at regular intervals (usually each minute) by lifting the occulting arm. In this way small well-defined images of Neptune in combination with distinct images of the satellite have been obtained, the photographs admitting of very accurate measurement of the position angle and distance of the satellite. The orientation was determined usually by means of a pair of short exposure images of Neptune, the clock being put out of gear for seven or ten seconds between the exposures to give a convenient displacement in R.A.

The earlier photographs from December 23 to January 19, ten in number, taken before the adaptation of the occulting shutter, were found not to admit of such accurate measurement, and they have therefore not been included in the series measured. The measures were made with a position-micrometer (formerly used for the measurement of solar photographs), which has been adapted by Mr. Simms to the measurement of position angles and distances on photographs of this class. The photographs were measured in reversed positions of the plate by each of two observers giving four independent sets of results. The mean values of position angle and distance as measured are given in the following table, the tabular positions being computed from the data given in the "Connaissance des Temps," based on Mr. H. Struve's elements, the eccentricity of the orbit being neglected owing to the uncertainty as to the present position of the periastron.

Positions of Neptune's Satellite measured on Photographs taken with the 26-inch Refractor.

Date.	Exposures.		Position Angle			Distance.		
			Observed.	Tabular.	Tab.-Obs.	Observed.	Tabular.	Tab.-Obs.
1899	d	h m	m	fs				
Jan. 26	9	40	20, 20	283°43	283°05	-0°38	14"02	14"20 +0"18
Feb. 2	7	37	20, 20	241°13	240°02	-1°11	16°15	16°07 -0°08
	17	7 15	17, 20	45°98	46°86	+0°88	14°59	14°51 -0°08
	28	9 46	15, 15	72°51	73°87	+1°36	15°82	16°50 +0°68
Mar. 1	8	35	20, 20	30°04	29°71	-0°33	12°42	12°44 +0°02
	1	9 44	20, 20	26°13	26°44	+0°31	12°07	12°12 +0°05
	2	9 9	20, 20	301°51	300°59	-0°92	11°78	12°01 +0°23
	5	8 47	15, 25	118°07	117°38	-0°69	12°08	12°28 +0°20
	9	8 44	20, 20	247°40	248°53	+1°13	15°96	16°34 +0°38
	10	8 41	25, 20	196°39	195°80	-0°59	10°51	11°21 +0°70
	14	8 55	15, 15	284°45	285°68	+1°23	12°89	13°49 +0°60
	27	9 24	20, 20	231°13	232°34	+1°21	14°70	14°82 +0°12

All the photographs were taken with the occulting shutter, except those on March 1. The orientation for the photographs on January 26, February 2 and 17 was determined from the tabular places of Neptune and a known star (at a distance of 32', 37' and 13' on the three dates respectively). On the other photographs it was found directly from the two images of Neptune displaced in R.A., as explained above.

The accuracy of the measures may be inferred from the following table, showing the discordances of the four independent sets of measures from the mean. The initials C. D. and P. M. are those of Mr. Davidson and Mr. Melotte, each of whom measured in the direct and reversed positions of the plate.

Satellite of Neptune. Discordances of Measures from Mean.

Date. 1899.	Position Angle.				Mean Discord.	Distance.				Mean Discord.
	C. D.		P. M.			C. D.		P. M.		
	D.	R.	D.	R.		D.	R.	D.	R.	
Jan. 26	+0°1	-0°5	+0°7	-0°3	±0°41	+0°2	+0°6	-0°5	-0°3	±0°4
Feb. 2	+0°1	-0°3	+0°4	-0°2	27	+0°6	-0°2	-0°9	+0°5	06
17	-0°3	+0°4	-0°2	+0°2	27	+0°11	-0°8	-0°8	+0°4	08
28	0°0	-0°1	+0°2	-0°1	11	+0°9	-0°12	-0°15	+0°16	13
Mar. 1	+0°7	-0°6	+0°4	+0°5	77	+0°1	+0°9	-0°7	-0°3	05
1	-0°1	-0°1	-0°3	+0°4	22	+0°10	-0°15	-0°18	+0°22	16
2	-0°2	-0°8	-0°1	+0°1	57	-0°7	0°0	-0°24	+0°31	16
5	-0°8	0°0	-0°3	+0°1	56	+0°13	+0°10	-0°14	-0°13	13
9	-0°1	+0°1	+0°1	-0°1	10	-0°2	+0°4	0°0	-0°4	03
10	-0°2	-0°5	+0°6	+0°1	37	+0°36	+0°15	-0°36	-0°15	26
14	-0°1	+0°1	0°0	0°0	05	+0°37	-0°14	-0°05	-0°18	19
27	-0°5	+0°1	-0°1	+0°5	±27	+0°16	+0°6	-0°49	+0°26	±24
Mean ±0°33						Mean ±0°13				

Notes.

Feb. 28, image of planet elongated; Mar. 1, occulting shutter not used; Mar. 2, image of satellite elongated; Mar. 5, satellite ill defined; Mar. 10, satellite within luminosity from Neptune.

Royal Observatory, Greenwich:
1899, May 12.

Observations of Swift's Comet, 1899, made at Grahamstown, South Africa. By Major L. A. Eddie.

The first news of the discovery of this comet by Swift reached us on March 10, but, owing to the prevailing cloudy skies of this very droughty season, I was unable to make a search for the comet till March 13, when a partial clearance of the sky permitted me to sweep for it. I soon picked it up in the 9½-inch reflector about 8 o'clock P.M. Cape uniform time. I found it fairly large and bright, of a very undefined outline, but considerably condensed in the centre, though showing no stellar nucleus or defined cometic envelopes. It was very fluffy and extremely ragged, with woolly protuberances on its northern edge, and possessed a faint, but long, straight tail, proceeding from an

arc about one-fifth of its periphery in the direction away from the sun. It shone with a bluish-white light. It was impossible under the unfavourable circumstances to make any exact estimate of its apparent size or length of tail, though this faint appendage could be dimly traced across the field of an eyepiece possessing a field 30' in diameter.

Friday, March 17. First evening since 13th inst. sufficiently clear of clouds for observation, and still only at intervals. Moon bright. Comet not visible to naked eye. In reflector the nucleus was more condensed than when last observed, and now nearly stellar. Coma very ragged and extended, and could be traced sweeping backwards in the well-known cometic form, and broadening the narrow tail observed on last occasion, which could still be traced for a distance about 30', while the visible head was about $3\frac{1}{2}'$ across.

Monday, March 20. Moon very bright. No great change perceptible, but cometic matter proceeding from the head, and then bending back and broadening the tail still more noticeably, also a faint northerly extension of tail.

Wednesday, March 22. Nucleus more condensed; now decidedly stellar. Coma very diffuse, but thin. Tail broader, and of a streaky, hair-like structure. Faint supplementary tail, diverging slightly to the north. On examination of nucleus with Espin ocular spectroscope it yielded a faint continuous spectrum crossed by three bright bands, evidently those of hydrocarbon; the central one, in the green, very broad, bright, and fluffy, ragged on both sides, but more so towards the blue; the second in brightness was situated about the greenish-yellow; and the third, a faint one, in the blue. When viewed with the McClean spectroscope, these bands were partially resolved into lines tapering off towards the more refrangible end.

Thursday, March 23. Moon very bright, and sky hazy. Seeing indifferent. The bending back of lateral streamers into tail very noticeable.

Friday, March 24. Observed comet in bright twilight. No apparent change. Nucleus stellar, head bright, tail broad and fairly well seen.

Saturday, March 25. Moon nearly full. Background too bright to examine detail.

Sunday, March 26. Moon full and very brilliant; but, notwithstanding, comet in reflector appeared bright with broad tail.

Monday, March 27. Dense cloud. Picked up comet for a moment only on side of field, then again obscured by cloud. Comet not since seen, being now lost in the twilight. The colour noted on each observation was bluish-white.

Approximate Positions.

	C.U.T. h m	R.A. h m s	S. Decl. ° ' "
March 13	8 5	1 22 45	- 15 40 "
17	8 0	1 7	- 11 11
20	8 0	0 59 25	8 23 30
22	7 15	0 55 25	6 29 30
23	7 15	0 53 22	5 32 30
24	6 4	0 51 10	4 44
25	6 25	0 49 31	3 54 30
26	6 30	0 41 40	3 5 30

Instrument $9\frac{1}{4}$ -inch reflector by Calver.
Powers 60 and 100.

Grahamstown,
1899 April 2.

Notes on the Spectra of γ Cassiopeiae and α Ceti.

By the Rev. Walter Sidgreaves, S.J.

 γ Cassiopeiae.

The photographs of the spectrum of γ Cassiopeiae obtained at this observatory are distributed over a period of eight years. There are fifty-two plates all told, of which half are by the old 8-inch glass and half by the Perry Memorial Objective of 15 inches aperture. With few exceptions they are all good photographs; but one of exceptional definition, of date 1898 March 7, was selected for the micrometer. From this plate the chart of the spectrum and table of wave-lengths were first constructed. The remaining plates were then examined, and it was found that with the guidance of the better plate nearly all the tabulated details could be traced also in the photographs of inferior definition.

It was not so easy to form a conclusion upon the general spectrum independently of the results of other observers. It might be in general a bright line or a dark line spectrum, but in either case it should be described as composed of hazy and weakly looking lines and bands. That there were some absorptions could hardly be doubted; the blue Hydrogen line H_γ appeared to be clearly resting on an absorption band; the Helium lines 4025 and 4471 and a line in the green 5295 were also absorptions. But for the rest, the condition of H_δ at one end of the spectrum and of the Magnesium group at the other were in favour of bright lines. The Hydrogen line seemed to have no

greater claim to be called bright than many of its neighbours ; and in the Magnesium group it was the silver deposit, not an intervening space, which fell to the wave-length 5170, which is the mean of the group. Influenced by these considerations, a table of wave-lengths was made out on the supposition of a bright line spectrum, and compared with the tabulated lines of other stars. It was then seen that many of the *Orion* lines, including all the Helium lines, agreed better with the spaces between the bright lines than with the lines tabulated. Another table was then made out for the lines, on the supposition of an absorption spectrum ; but it was found impossible in this operation to avoid tabulating many bright lines, and the result obtained is a mixed spectrum of bright and dark lines.

The General Spectrum.—The complete spectrum of the star is given in tabular form, with the bright lines indicated by the letter *b* written after them ; and in a parallel column the absorption lines of γ *Orionis*, as measured on a plate of date 1896 December 28, are entered for comparison. It will be observed that both dark and bright lines of γ *Cassiopeie* are fairly well matched by *Orion* lines. An asterisk (*) means that the line is near an *Orion* line not seen in γ *Orionis*. A dagger (†) means that the line is not contained in Pickering's list of *Orion* lines.

The Hydrogen Spectrum.—The following figures, collected from the tabulations of the general spectrum, exhibit the Hydrogen spectrum separately, with the character and relative intensities of the lines :—

	He.		H δ .		H γ .		H β .	
	<i>i</i>		<i>i</i>		<i>i</i>		<i>i</i>	
Absorption	3963	6-	4095	5-	4334	4-		
Radiation	3970	0	4101	2	4340	8	4861	10.
Absorption	3976	6-	4107	5-	4347	4-		

The extreme figures in each case indicate, in wave-lengths, the margins of the broad absorption bands. It will be noticed that the intensity assigned to the radiation H ϵ is zero. This means that the silver deposit is about the same as that of the continuous spectrum, and the line appears by contrast on the broad absorption. H δ is stronger than the continuous spectrum, and H γ much more so. All three have the same appearance of a reversal in the centre of a broad absorption line ; but H β is superlatively bright, with only a very weak, if any, absorption background. The radiation or bright lines increase in intensity with increasing wave-length, while the absorptions fall off. The Hydrogen spectrum therefore of γ *Cassiopeie* is very closely the same as described by Professor Pickering in the "Harvard College Annals" in 1897,* and differs from Sir Norman Lockyer's photographs only in the appearance of H β , which on the South Ken-

* XXVIII. i. 100.

sington plates appears superposed on a broad dark band.* And this difference is probably due to the smaller dispersion employed at Stonyhurst. On some plates by greater dispersion the dark H_β is well marked, but not strongly.

That the bright Hydrogen lines are doubles has been shown by Lockyer, Newall, and McClean. They are not represented so either on the map or in the table of wave-lengths, both of which were constructed from a photograph too small to show the separation. On other plates with the greater dispersion of two compound prisms H_γ has the convincing appearance of a double, inasmuch as it appears not stronger at the centre than at the margins of a comparatively broad line; and on one plate the middle of the line is decidedly weaker.

The Helium Spectrum.—In the following table the Helium lines, as published by Runge and Paschen,† are collated with absorption lines in γ Cassiopeiae:—

Helium. λ	i	γ Cassiopeiae. λ	i	Helium. λ	i	γ Cassiopeiae. λ	i
4009	1	4009	5	4438	1		
4026	5	4025	6	4471	6	4471	6
4121	3	4118	3	4713	3	4711	1
4144	2	4144	4	4922	4		
4169	1	4170	3	5016	6		
4388	3	4388	4	5048	2	5048	2

The Bright Lines.—The supposed Magnesium group at 5170 as a bright line group is confirmed by the high temperature Magnesium line at 4481 appearing also as a bright line. This cannot well be taken for the Iron line 4481.6, because it would be the only strong Iron line in the spectrum of the star out of many equally strong lines in the Iron spectrum. Sherman‡ has the 5170 line at 516.75, which is a nearer match to the head of the Carbon band at 5165 than our line at 5170, but it is quite impossible to reduce the wave-length we have tabulated.

The lines 4518 and 4586 are those noted by Pickering as examples of "bright regions which are not well defined bright bands, . . . but have rather the nature of bright spaces between dark lines."§ Our judgment of them is strongly in favour of true bright lines. The longer wave-length of the two is the brightest line in the spectrum after H_γ , and on some plates its appearance suggests that it would resemble H_γ very closely if it were set off on a similar absorption background.

The line at 5020 is the line noted by Pickering at 5023 as the strongest of the bright lines not due to Hydrogen, and he does not speak of bright 5170; on our plates these two lines are of equal bright intensity.

* *Proceedings of the Royal Society*, lvii. 176.

† *Astrophysical Journal*, iii. 10.

‡ *Astr. Nach.* No. 2707.

§ *Harvard College Annals*, XXVIII. i. 101.

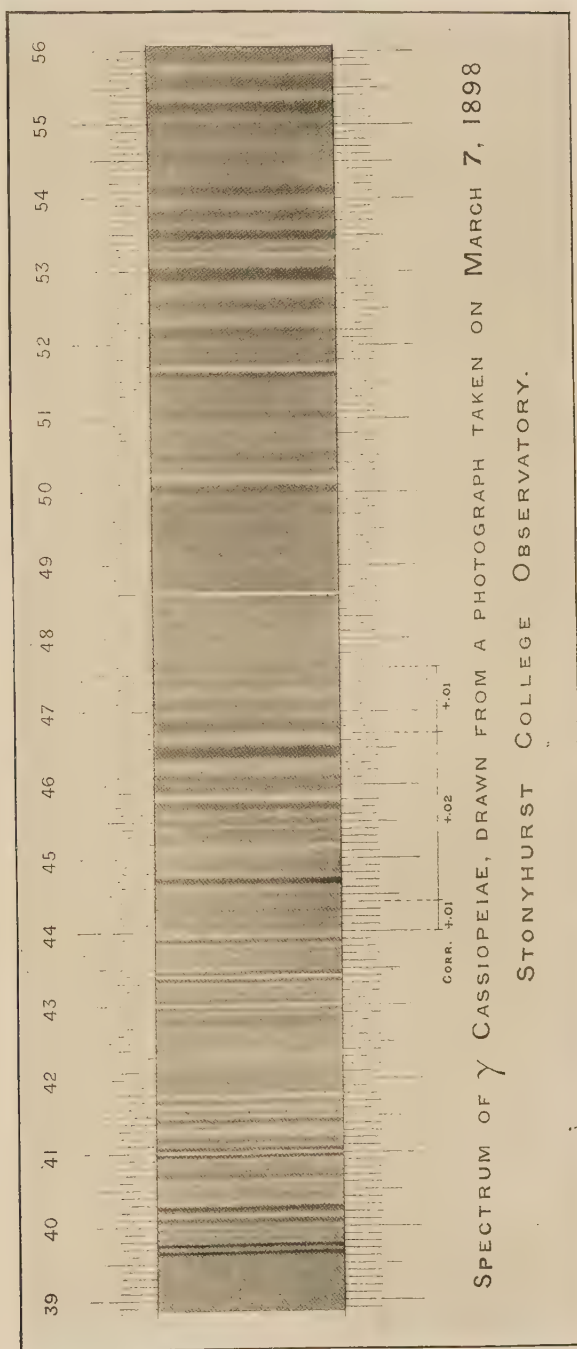
Origins.—Excepting Hydrogen, Helium, and Magnesium, there are no very probable origins assignable. The strongest bright line, already mentioned at 4586, might be attributed to Vanadium, if the stronger lines of the metal made a better figure in the spectrum of the star; but the lines that should be the strongest are the weakest. The following table shows how they are matched in an inverse order of their intensities :—

Bright Lines. γ Cassiopeæ. λ	<i>i</i>	Vanadium (Thalén). λ	<i>i</i>
4382	1	{ 4379.0	10
		{ 4384.1	10
4395	2	4395.1	6
4586	6	4585.1	4

Variability.—The Hydrogen spectrum has shown no signs of alteration during the eight years of observation, but in other parts of the spectrum there is a probability of changes. These in general are not sufficiently pronounced to safeguard our conclusions against erroneous changes, which may be due only to atmospheric effects upon the photographic definition. But the strong bright (Vanadium?) line 4586 appears on several plates with a greater probability of real change. On some plates it is a clear single strong line; on others its appearance is that of a double, too close for separation by small dispersion and of less intensity, with the apparent widening all on the side of shorter wave-length. Real change may account for this line escaping the notice of Sherman at Yale College, and for its appearance on the Harvard College plates as inferior to the line 5023. Our measure of this latter line is 5020, the same as Sherman's dark line 502. Its brightness is noted as varying between 1 and 4, which may be attributed to photographic imperfections; but as a dark line recorded by Sherman, and as a bright line on the photographic plates at Harvard College and Stonyhurst, it claims to be a variable.

The following comparisons with Sherman's lines will serve to show where possible changes may be looked for, bright lines and dark lines being distinguished by the letters *b* and *d*. But the unlettered figure 5275 represents a space between absorption lines :—

Yale.	Stonyhurst.	Yale.	Stonyhurst. Mean.
399.3 <i>d</i>	3995 <i>d</i>		5275
418 <i>b</i>	4177 <i>b</i>	530.98 <i>b</i>	5316 <i>b</i> } 5296
462.3 <i>b</i>	4628 <i>b</i>		
467.3 <i>d</i>	4681 <i>d</i>	542.2 <i>b</i>	
492 <i>d</i>	4711 <i>d</i>		
499 <i>b</i>			
502 <i>d</i>	5020 <i>b</i>	555.75 <i>b</i>	5540 <i>b</i> } 5558
516.75 <i>b</i>	5170 <i>b</i>		5576 <i>b</i> }



α Ceti.

During the recent period of maximum brightness of α Ceti in the autumn of 1898, the weather was far from favourable, and only seven out of thirteen exposures gave good photographic spectra of the star. These are stronger than the photographs obtained in the previous maximum period, owing to the greater magnitude attained in 1898. The spectrum is the same in all its details, with the single exception that the possible bright line at 4862 appears in much stronger contrast than in 1897, and under conditions of defective definition which would then have completely obliterated it. But apart from its position of the H_{β} radiation, it could not be called a bright line. H_{γ} and H_{δ} retain their extraordinary brilliancy, with a possible increase of difference between the two intensities in favour of H_{γ} .

A wave-length correction which affects the tabulations and chart of the spectrum of α Ceti, as given at pages 346-352, vol. lviii. *Monthly Notices*, between $\lambda\lambda$ 4400 and 4760, has been found necessary. The conclusion was drawn from the comparison columns of γ Cassiopeiae and γ Orionis. In both columns the Helium line 4471 appeared at 4469. Six other Orion star spectra were then measured, and each gave the line at 4469. But two of these had been also photographed with a two prism dispersion, and these gave the line at 4471. The two-prism wave-length-curve was then carefully re-examined with the aid of Dr. Scheiner's tabulated solar lines in a *Aurige* and Rowland's map of the solar spectrum. A large number of these lines were satisfactorily identified on the two-prism photograph of the spectrum of *Arcturus*, and were found, by the curve, to agree with Scheiner's figures to nearly the fifth figure. The shorter, or one-prism curve, was then corrected by the longer one, through the medium of the same star spectra as given by the one-prism and by the two-prism dispersions. The spectra employed were of α Boötis, η Ursæ Majoris, and α Cygni. The resulting corrections* are to the fourth figure—

+ 1 between 4400 and 4440

+ 2 „ 4440 „ 4670

+ 1 „ 4670 „ 4760

These corrections, applied to the tabulations of α Ceti from the photographs of 1897, improve their relations to the strong metallic

* This correction is applicable to the tabulations of β Lyræ, 1895, but not to those of 1893, nor to those of the *Nova*, 1892, which were by another instrument.

lines; and the middle subdivision of the band which begins at λ 446 is brought nearer to the strong Helium line 4471.6.

The Hydrogen Spectrum of α Ceti and of γ Cassiopeiae.—The bright Hydrogen lines of γ Cassiopeiae seem to exhibit the same character of radiation as that of the electrified Hydrogen tube of the laboratory. In both, the density of the silver deposit upon the photographic plate is greatest by H_{β} radiation and decreases with the shorter wave-lengths. The two brilliant lines of α Ceti, H_{γ} and H_{δ} , do not appear to follow the same law, H_{δ} being far too strong compared with H_{γ} ; but this may be accounted for by the greater strength of the continuous spectrum about H_{γ} showing less contrast. And this explanation is made more probable by a smaller photograph with a single half-prism, in which the H_{γ} line is almost lost in the condensed continuous spectrum of its neighbourhood. If we suppose the relative intensities of the lines of α Ceti as they appear on the plate without reference to the sensibility curve to be the same as found in the laboratory, it is not easy to imagine the condition of things able to stop out so powerful a radiation as that of H_{β} . Another query suggests itself: Where is the absorbing atmosphere capable of stopping out H_{ϵ} , H_{β} and H_{α} ? Dr. Scheiner's suggestion for γ Cassiopeiae, of a Hydrogen atmosphere great compared with the photosphere of the star, is the only one which seems reasonable, if we limit our thoughts to our present knowledge of spectroscopic phenomena. The explanation fits the observations of γ Cassiopeiae very satisfactorily. The denser Hydrogen near to the photosphere would give the broad dark bands, and the rarefied Hydrogen more remote from the centre would give the bright reversal in the middle. The other lines, bright and dark, would respectively indicate elements *diffused* through the Hydrogen atmosphere, and others *low-lying only*; a supposition well in keeping with the appearance of the lines, which is that of a struggle between radiation and absorption to impress its mark on the plate. But when applied to α Ceti the explanation only adds new puzzles to the already perplexing spectrum of the star. In this spectrum we have Hydrogen radiation far more intense than in γ Cassiopeiae. We have ink-black broad absorptions, and a brilliant photospheric spectrum. To account for the glowing Hydrogen lines we have to suppose either a far more vast Hydrogen atmosphere or a greatly higher temperature; and for the missing lines we need dense vapours at lower temperatures and co-extensive with the Hydrogen atmosphere. It seems preferable to go beyond our laboratory knowledge and suppose an abnormal Hydrogen radiation of a high degree of energy in which some of the oscillation frequencies have fallen out of the spectrum.

May 1899.

of γ Cassiopeiae and α Ceti.

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γ Cassiopeiae. λ i	γ Orionis. λ i	γ Cassiopeiae. λ i	γ Orionis. λ i
[3963 6]		4395 <i>b</i> 2	4393 I
3970 <i>b</i> 0	3970 10		4399 I
[3976 6]			4409 I
			4418 2
	3980 I	4431 3	4428 I
3983 I	3985 I		4439 2 <i>d</i>
3995 2	3995 I	†4451 I	
4009 5	4008 7		4457 I <i>d</i>
	4016 2	†4462 <i>b</i> I	
4025 6	4026 9	4471 6	4471 6
	4033 I		4478 I <i>d</i>
*4037 I		4481 <i>b</i> 2	
4043 I	4043 I <i>d</i>	†4485 I	
	4053 I		4492 I <i>d</i>
	4059 I		4506 I
4069 3	4068 I	4518 <i>b</i> 3	4517 I
	4072 I		4532 I
4076 2	4076 I		4543 I
	4085 I		4552 2
4088 I	4090 I <i>d</i>	4573 4 <i>w</i>	4569 2
			4579 I
[4095 5]		4586 <i>b</i> 6 <i>w</i>	
4101 <i>b</i> 2	4097 2		4593 I
[4107 5]	4101 10	*4596 3	4606 I
		*4612 3	
	4111 I		4621 I
4118 3	4121 5	4628 <i>b</i> 4 <i>w</i>	4632 I
	4129 2		4643 2
4131 <i>b</i> 3	4131 I	4647 6 <i>w</i>	4652 2
	4135 I	4664 <i>b</i> 4 <i>w</i>	4664 I
4144 4	4145 7	4681 3 <i>w</i>	4678 I <i>d</i>
4155 I	4153 I <i>d</i>		4697 2
	4162 I		4705 I
4170 3	4169 3	4711 I	4713 2
†4177 <i>b</i> 2			4731 I
4185 2	4186 I		4738 I
	4198 I		4746 I
	4204 I		4760 2
	4211 I		4782 I

γ Cassiopeiae. λ	i	γ Orionis. λ	i	γ Cassiopeiae. λ	i	γ Orionis. λ	i
		4225	1			4796	2
†4234 <i>b</i>	2					4821	1
4239	1 <i>d</i>	4238	1			4839	2 <i>d</i>
		4248	1	4861 <i>b</i>	10	4861	10
4253	1	4255	1			4871	2
4266	1	4268	3			4885	3
		4275	1			4902	3
4281	2 <i>w</i>	†282	1			4923	6
		4289	1	5006	3		
4295	2	4295	1	5020 <i>b</i>	4 <i>d</i>		
4302 <i>b</i>	2			5048	2		
4306	2	4304	1	5104	2		
4317	1	4317	1	5160	3		
		4323	1	5170 <i>b</i>	4		
4326	1	4328		5214	3 <i>w</i>		
				5256	3 <i>w</i>	5254	5
4334	4			5295	4 <i>w</i>		
4340 <i>b</i>	8	4340	10	5316 <i>b</i>	2		
4347	4			5350	4		
		4353	1 <i>d</i>	5376	3		
		4364	2	5411	3		
		4374	1	5524	3		
*4382 <i>b</i>	1			5540 <i>b</i>	2		
4388	4	4388	5	5576 <i>b</i>	2		

b = bright line.*w* = wide line.*d* = double line.

The region beyond λ 4923 of γ *Orionis* has not yet been completely mapped.

Stonyhurst College Observatory.

Longitude from Moon Culminations. By D. A. Pio.

(Communicated by the Secretaries.)

A. Theory.

Purpose of the New Method.—For the determination of longitude on land, especially during journeys through the continents of Africa and Australia, as also on touching an unknown island in the southern seas, the author dares to express his hope that the method put forward in this paper is preferable to the obsolete one of lunar distances.

In the new method the culmination of the Moon serves only to determine exactly, easily, and quickly the precise instant of the Moon's passage through the local meridian. The longitude is deduced from this instant as in the method of Moon's transits. The practical advantage of the new method is that it does not require any transit instruments, and dispenses with the laborious setting of a telescope exactly in the meridian.

The observer is supposed to possess a good sextant furnished with a telescope magnifying at least twice, an artificial horizon, a *well-rated* chronometer, the *Nautical Almanac* for the current year, a set of nautical and logarithmic tables, and last, but not least, to operate on good solid ground, with an assistant to note down the instants of the different observations.

Remarks on Culminations.—The author would draw the attention of the reader to the sense of the word "culmination" as used in this paper. The author does *not* mean by it that the centre of the celestial body is exactly on the local meridian, but that this body is at its greatest altitude. Whether the greatest altitude coincides or not with the meridian passage is quite another question. The author reminds the reader that only the fixed stars culminate really in the meridian. The Sun, the Moon, and all the planets culminate *out* of the meridian.

"Meridian passage" or "transit" is the word the author uses in this paper instead of "culmination," when he means that the centre of the celestial body is exactly on the meridian.

Principle of the New Method.—The local longitude is found very simply through the right ascension of the Moon at the instant of her transit at the place of observation. This right ascension furnishes the corresponding mean time of Greenwich, and the difference between this time and the mean local time at Moon's transit is the longitude sought, in time. So far, there is nothing new.

In the new method the right ascension is not found directly by observation, but deduced from the mean local time of Moon's meridian passage. For this time converted into sidereal time and added to the right ascension of the mean Sun at his transit

in the place of observation, gives immediately the sought right ascension of the Moon.

The real difficulty, therefore, is to find exactly the mean local time at Moon's transit without using a transit instrument. The culmination of the Moon out of the meridian is substituted for her meridian passage, and the lapse of time between transit and culmination is found by a calculation which the author calls "reduction to the meridian." Thus, the instant of Moon's culmination gives the instant of Moon's transit.

The culmination of the Moon can be observed with a good sextant, and so the heavy transit instrument and the difficult operations required to place the instrument exactly in the meridian are dispensed with. However, the culmination can *not* be observed directly, and the author deduces the exact instant of meridian passage from the instants at which two equal altitudes of the Moon are taken. The middle time between these two instants is used in the calculation of reduction to the meridian, instead of the precise instant of culmination.

The instant of Moon's meridian passage becomes therefore known when the instants of two equal altitudes are given in mean local time. This local time requires the determination of an hour angle from an altitude of the Sun, for the observer possesses only a chronometer indicating, more or less exactly, mean Greenwich time.

This hour angle can be dispensed with, and all errors consequent upon it can be totally eliminated by simply determining the instant of Sun's culmination in the place of observation. This is done, as above in the case of the Moon, by taking two equal altitudes of the Sun. The middle gives then, by reduction to the meridian, the time of Sun's meridian passage, which must be corrected by the equation of time in order to give the instant of mean noon at the place of observation.

When the instants of Moon's transit and of mean Sun's transit have been deduced by calculation from the instants, as given by the chronometer, of the equal altitudes both of Sun and Moon, the mean local time of Moon's transit is given by subtraction.

The novelty in the new method consists, therefore, in the use of the method of equal altitudes. This use renders unnecessary the determination of the meridian's position, the employment of complicated instruments, the observation of Moon culminating stars, and the calculation of hour angles. The arcs measured with the sextant are not used at all in the calculation. All is reduced to the indications of the chronometer.

The practical difficulty in this method consists only in the perfect determination of the instants at which the different equal altitudes are taken.

How to Make the Observations Precise.—In order to obtain the longitude with accuracy the lapse of time between Sun's

transit and Moon's transit must be correct down to the tenth of a second, as even this small fraction corresponds to a mean error in longitude of two-thirds of a minute (of arc). The first requirement is, therefore, that the rate of the chronometer be as uniform as possible, and that its daily amount be ascertained with the greatest precision. In consequence the two culminations of the Sun and the Moon ought to be as near to each other as possible, and to be chosen so that the interval between them be *never* more than twelve hours.

The instants of the different observations must be given to tenths of seconds, which can be done very easily by the chronograph. However, the difficulty is not in noting down the instant of observation, but in catching the precise instant at which the Sun or Moon is *exactly* at the altitude indicated by the graduation. In order to facilitate this the observation must be made with the artificial horizon, and the telescope of the sextant must have a certain magnifying power—say, about five. In this way the contact of the direct image with the reflected one can be observed with more distinctness, and the real instant of contact is perceived with fewer chances of error.

Every error in the instant of observation will influence the middle time deduced therefrom, and in order to give to these times a greater degree of exactness four pairs of equal altitudes must be taken for the Sun and as many for the Moon. The mean of the four middle times for the Sun (or the Moon) may be trustworthy.

Besides, the altitudes of both Sun and Moon must be such that a perceptible change in altitude takes place in so small an interval of time as a tenth of a second. Therefore, the altitudes must be low, but not so much that variations in refraction may make unequal altitudes appear equal.

For this reason the new method cannot be used with great success in high latitudes.

Reduction to the Meridian.—The lapse of time between meridian passage and middle time is given by a well-known formula, which all treatises on nautical astronomy bring under the head 'time from equal altitudes.' The author uses this formula under the following form :

$$x'' = + \frac{\delta_1'' - \delta_2''}{2 \sin H} \left\{ \cot \frac{\delta_1 + \delta_2}{2} \cos H - \tan \lambda \right\},$$

where x'' is the angle in seconds at the pole between the meridian and that circle of declination which bisects the angle $2H$;

$2H$ the angle at the pole between the circle of declination of the Sun or the Moon at the first observation, and that of the same celestial body at the second observation ;

δ_1 the polar distance from the elevated pole of the observed celestial body at first observation ;

δ_2 the polar distance of the same body at second observation ;

λ the latitude of the place of observation ;

$\delta_1'' - \delta_2''$ is the difference in seconds between the polar distances.

The angle α'' must now be divided either by $15''$ in the case of the Sun, or by $15'' \cdot 0411 - \frac{\Delta^s}{40}$ in the case of the Moon, in order

to give the lapse of time between meridian passage and middle time. The symbol Δ^s stands for the variation in 10^m of the Moon's right ascension, as given by the *Nautical Almanac* for the Greenwich time at Moon's transit.

The difference $\frac{\delta_1'' - \delta_2''}{2}$ must be calculated with the greatest precision, as any error in it has an influence on the right ascension of the Moon from which the longitude is obtained.

The angle $2H$ is deduced from the interval of time between the observations of the two equal altitudes. $2H =$ this interval in seconds, multiplied by $15''$ in the case of the Sun, or multiplied by $15'' \cdot 0411 - \frac{\Delta^s}{40}$ in the case of the Moon.

Error in Longitude.—As the variation in 10^m of Moon's right ascension oscillates from 17^s to 29^s , the error in longitude corresponding to one second of right ascension oscillates from $\frac{150'}{17} = 8' \cdot 8$ to $\frac{150'}{29} = 5' \cdot 2$, the mean being $7'$. The sources of error are :

1. Wrong middle time, arising from want of correspondence between the altitudes measured by the sextant and the indications of the chronometer.
2. Wrong reduction to the meridian arising from wrong values of latitude and polar distances being used in the calculation.
3. Wrong lapse of time between the transit of the Sun and that of the Moon, arising from wrong value of chronometer's daily rate.

The author makes thereon the following remarks :

To 1. The use of four pairs of equal altitudes combined with the employment of the artificial horizon and the magnifying power of five renders the mean of the middle times exact to one-tenth of a second at least.

To 2. When the quantity $\frac{\delta_1'' - \delta_2''}{2}$ is calculated exactly, and that can always be done, the wrong values of latitude and of the polar distances have no influence on the reduction to the meridian.

To 3. Nowadays chronometers are so well constructed that the changes in the daily rate are insignificant ; besides, the influence of temperature is now taken into account.

Therefore the author hopes that the error in longitude by the new method will amount only to a few minutes of arc in unfavourable circumstances.

Longitude by the Sun.—As in the new method the instant of the mean Sun's transit at the place of observation is given in mean Greenwich time, the local longitude in time is evidently the difference between 12^h and this Greenwich time.

Therefore the longitude is furnished by the new method in two different ways :

1. Longitude by the Sun, deduced directly from the chronometer ;
2. Longitude by the Moon, deduced from her right ascension.

Comparison of the new method with the known ones.—The author has occupied himself during many years with finding out a new method for the determination of longitude and tried many contrivances of his own. He has come to the following conclusions :

1. On sea the *best* method is to determine longitude by the chronometer compared with local time deduced from altitudes of the Sun.
2. On sea the method of lunar distances is still the *best* when one wishes absolutely to use the Moon for determining the longitude on board a ship. All methods proposed to supersede Borda's one are inferior to it, and it is lost time and wasted ingenuity to devise such new methods.
3. On sea Borda's method cannot compete with longitude by chronometer, especially now that chronometers have so much improved.
4. On land the telegraph is unsurpassable in its accuracy for the purpose of determining longitudes.
5. On land the method of deducing the right ascension of the Moon from comparison with moon-culminating stars comes the next after the telegraphic method. Its inconveniences consist :
 1. In the necessity of determining accurately the local meridian.
 2. In the use of heavy instruments.
 3. The delicacy of the observations, and the many corrections required.
6. The new method requires no meridian, may be used very easily and everywhere on land, and rests on the indications of the chronometer only. The measurements with the sextant are only used to fix the instants of the observations. The new method is properly a chronometrical method.

7. On land the longitude by the Sun, as determined by the new method from equal altitudes of the Sun, is, when a well-rated chronometer is used, the simplest of all methods after the telegraphic one.
8. The longitude by the Moon, according to the new method, is a good check on the chronometer.

B. Practice.

Directions for the observer.—(1). Do not use this method when your latitude exceeds sixty degrees north or south. (2) Determine your latitude with the greatest precision by the artificial horizon, if possible, by the means of different observations. (3) Choose such a day that the Moon's daily variation in right ascension be as great as possible. (4) With the best assumed longitude calculate the approximate Greenwich time of Moon's meridian transit. (5) Before culmination take four altitudes of the Moon in succession, from three to three minutes, and as many corresponding ones after culmination. (6) The observed altitudes of the Moon must not be less than thirty degrees, nor more than sixty. (7) The time of taking the altitudes of the Moon must not be less than one hour and a half before and after culmination, nor more than two hours before and after it. (8) For the nearest Sun's meridian transit take four successive altitudes of the Sun before noon, as near the prime vertical as possible, and as many corresponding ones after noon. (9) All altitudes must be taken with the artificial horizon. (10) The instant of every observation must be taken with a chronograph to tenths of seconds.

Rules for the calculations.—(1) First calculate longitude by the Sun, then longitude by the Moon. (NOTE—These two calculations are to be effected in the same manner, except where the rules expressly indicate the contrary.) (2) Calculate the mean time of the four observations before culmination and that of the corresponding ones after culmination. From these two means deduce the middle time. (3) Find the difference between these two means and convert it into seconds. In the case of the Sun multiply the half of the found number of seconds by 15 and call the result H. (3 *bis*) In the case of the Moon find first in the *Nautical Almanac* the variation in 10^m of the Moon's right ascension for the hour of middle time, divide this number by 40 and subtract the quotient from $15^{\prime\prime}0411$. The result is the factor of the half difference between the means. (4) Calculate (down to tenths of seconds) the polar distance from the elevated pole of the Sun (or Moon) for the instant given by the mean of the observations before culmination and call it δ_1 . (5) Calculate (down to tenths of seconds) the polar distance from the elevated pole of the Sun (or Moon) for the instant given by the mean of

the observations *after* culmination and call it δ_2 . (6) Take the half *sum* of these two polar distances and call the result $\frac{\delta_1 + \delta_2}{2}$.

(7) Take the half *difference* of these two polar distances, express the result in seconds, and call it $\frac{\delta_1'' - \delta_2''}{2}$. (NOTE I.—In the case

of the Moon it is necessary to verify the amount $\frac{\delta_1'' - \delta_2''}{2}$ by

calculating separately the variation of Moon's declination during the interval of time elapsing from the mean instant of the observations before culmination to the mean instant of the observations after culmination. NOTE II.—Particular attention must be paid to the *sign* of $\frac{\delta_1'' - \delta_2''}{2}$.) (8) For longitude by the

Sun calculate the equation of time for the instant of Sun's transit at the place of observation, with the help of the longitude by account. (9) Calculate the reduction to the meridian by the rules in Article 11 and pay particular attention to the *sign* of the result. (10) To the middle time, as found by Rule 2, apply the above reduction to the meridian with its *proper* sign. (NOTE—In the case of the Sun apply also the equation of time with its *proper* sign.) (11) For longitude by the Sun call the result 'Greenwich time of mean Sun's transit,' and find the difference between it and 12^h. This difference is the sought longitude. (12) For longitude by the Moon call the sum of middle time and reduction to the meridian 'Greenwich time of Moon's transit,' and find the difference between it and that of mean Sun's transit. The result is the mean local time of Moon's transit. Mark whether it is a.m. or p.m. (13) Convert the local time of Moon's transit into the equivalent sidereal time (down to hundredths of seconds) and call the result 'Moon's difference in right ascension.' (14) Calculate (down to hundredths of seconds) the right ascension of the mean Sun at the Greenwich time of mean Sun's transit in the place of observation. (15) To this right ascension of the mean Sun *add* the sidereal time calculated by Rule 13, if the local time of Moon's transit is p.m.; *subtract* this sidereal time from the above said right ascension of the mean Sun when the said local time is a.m. The result is the Moon's right ascension at the instant of Moon's transit. (16) Calculate (with the help of the *Nautical Almanac*) the Greenwich time corresponding to this right ascension of the Moon. The difference between this Greenwich time and the local time of Moon's transit is the sought longitude in time.

Rules for reduction to the meridian.—(1) Take the logarithm of cosine H, that of cotangent $\frac{\delta_1 + \delta_2}{2}$, add them, and find the *number* corresponding to the sum. If $\frac{\delta_1 + \delta_2}{2}$ is less than 90° this number is positive; if it is more than 90°, negative.

(2) From this number subtract *algebraically* the *natural* tangent of latitude and give to the result its *proper* sign. (3) Take the logarithm of this result, that of $\frac{\delta_1'' - \delta_2''}{2}$ as expressed in seconds, the arithmetical complement of sine H and the arithmetical complement of $15''$ for the Sun, or that of $15'' \cdot 0411 - \frac{\Delta^s}{40}$ for the Moon, add all together and find the *number* corresponding to the sum. This number is the reduction to the meridian in seconds of time. Pay particular attention to its time. (NOTE I.—It is good to write the words ‘positive’ or ‘negative’ opposite the different numbers entering into this calculation. NOTE II.—For the Sun *four* decimal places are sufficient for the abovesaid logarithms. For the Moon *five* decimals are necessary.)

Example.

1885 March 27, in a place on the island of Syra, situated exactly in latitude $37^\circ 25' 30''$ N., but in longitude by account 25° E., the following observations were taken with the artificial horizon :

Measured Heights.				Instants of Observation.			
					h	m	s
1. Sun's L.L.	...	...	78 36 0	...	7	34	50.9 a.m.
2. „	...	...	78 36 0	...	0	55	10.7 p.m.
3. Moon's L.L.	...	...	107 44 20	...	6	34	33.4 p.m.
4. „	...	...	107 44 20	...	9	37	36.2 p.m.

The above times are in Greenwich time, as given by a chronometer whose indications have been corrected for error and daily rate.

Middle time	=	h	m	s		o	'	''
		10	15	0.8 a.m.				
Half difference (= H)	=	2	40	9.9	=	40	2	28.5
Polar distance of Sun at first observation (= δ_1)	=				=	87	17	39.2
Polar distance of sun at second observation (= δ_2)	=				=	87	12	26.5
$\frac{\delta_1 + \delta_2}{2}$	=				=	87	15	02.9
$\frac{\delta_1 - \delta_2}{2}$	=				=	2	36	35

Reduction to the meridian calculated by above
formula = - 11.8

Middle time	...	...	...	...	...	h m s	10 15 00.8 a.m.
Reduction to the meridian	...	...	...	...	...		- 11.8
Equation of time	...	...	...	...	...		5 23.0
Greenwich time of mean Sun transit	...	...	...	...	...	h m s	10 20 12.0 a.m.
Local time	...	...	...	...	...		12 0 0.0
Difference	...	...	...	...	...		- 1 39 48
Longitude of Syra by the Sun	...	...	...	...	...	° ' "	24 57 0.0

CALCULATION OF LONGITUDE BY THE MOON.

I. Preliminary Calculations.

Time of Moon's first observation	...	...	...	h m s	6 34 33.4 p.m.
Time of Moon's second observation	...	...	...		9 37 36.2 "
Middle time	...	...	=	h m s	8 06 04.8 p.m.
Half difference	...	...	=		22 5 8
Polar distance of Moon at first observation (= δ_1)	...	...	=		81 27 51.5
Polar distance at second observation (= δ_2)	...	...	=		81 57 32.8
$\frac{\delta_1 + \delta_2}{2}$	...	...	=		81 42 42.2
$\frac{\delta_1 - \delta_2}{2}$	...	...	=		- 14 50.65

Reduction to the meridian calculated by the
above formula ... = 1^m 43^s.14

Middle time	...	...	...	h m s	8 6 04.80 p.m.
Reduction to the meridian	...	...	...		1 43.14
Moon's transit in Syra (Greenwich time)	...	...	...	h m s	8 7 47.94 p.m.
Mean Sun's transit in Syra	...	...	...		10 20 12.00 a.m.
Moon's transit in Syra (Syra time)	...	...	...	h m s	9 47 35.94 p.m.
Acceleration for 9 ^h 47 ^m 36 ^s	...	...	...		1 36.53
Sidereal time at Sun's transit in Syra	...	...	...		20 3.05
Moon's right ascension at transit at Syra	...	...	...	h m s	10 9 15.52

II. Calculation of Longitude.

Moon's right ascension for March 27, 8 ^h is	...	...	...	h m s	10 8 57.88
Change in 7 ^m 50 ^s .4 is	...	...	...		17.64
Therefore at Greenwich time 8 ^h 07 ^m 50 ^s .4 the Moon's R.A. is	...	...	...	h m s	10 9 15.52

							h	m	s
Moon's transit (Syra time) ...	...	...	...	...	...	9	47	35	9 p.m.
„ (Greenwich time) ...	...	...	...	...	...	8	7	50	4 p.m.
Difference ...	...	...	...	...	...	1	39	45	5
Longitude of Syra by Moon...	...	...	...	...	...	24	56	22	5 E.

Syra, Greece:
1898 November 2.

Errata.

Professor Schur's paper on *The Diameter and Compression of the Planet Mars*.

Page 330, first line of table, mean of h and v , for $6''\cdot23$ read $6''\cdot28$.

„ 331, third line from bottom, should read $2a$, $6\cdot370$; $2b$, $6\cdot275$; Diff.

$$0\cdot095; a = \frac{a-b}{a}, 67$$

Cape Observations of Nebulæ: Page 339, the paragraph commencing h 262 should run on with next paragraph, and read—

. . . and is $13' N. p$ h 2630 = G.C. 838 . . .

Cape Observations of Occultations, page 340, lines 11, 12—

7-in. equatorial, δ lat., for — read +.

10-in. astrographic telescope, δ lat., for + read —.

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